BOOK REVIEWS

This issue marks a changing of the guard. After fifteen years as editor of book reviews, Bob O'Malley is stepping down. I am sure you will all agree with me that Bob has done a tremendous job, and I will admit that I feel some trepidation as I attempt to step into his shoes. At the same time I am excited for the opportunity to serve SIAM in this capacity. I have put together an editorial board to help me with the task; I thank Michele Benzi, Krešo Josić, Hinke Osinga, Nick Trefethen, and Thaleia Zariphopoulou for agreeing to work with me.

The book reviews in this issue were assembled by Bob; soon it will be my turn. The featured review by Andreas Albrecht takes a detailed look at Max Tegmark's quest for the ultimate nature of physical reality. He recommends the book and even suggests that it would be of value to readers who are not technically oriented. He declares himself unready to jump on Tegmark's bandwagon but finds that the book was well worth reading nevertheless. Each physicist defines boundaries for himself (or herself) and walks a line between being too conservative and being a crackpot. Albrecht writes, "Reading this book has caused me to stop and reflect on my own boundaries and the ideas that create structure for my own research."

As usual we have reviews on a wide variety of topics. The book *Introduction to Global Optimization Exploiting Space-Filling Curves* by Sergeyev, Strongin, and Lera, reviewed by Sergiy Butenko, particularly caught my eye. I would never have thought that space-filling curves could be practical tools for global optimization. I can see that I am going to learn a lot in this new job.

The issue concludes with two reviews by Bob O'Malley himself, one on a collection of recollections about the brilliant ex-mathematician Alexander Grothendieck, the other on a different sort of collection: views of mathematicians on the subject of creativity.

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Book Reviews

Edited by David S. Watkins

Featured Review: Our Mathematical Universe: My Quest for the Ultimate Nature of Reality. *By Max Tegmark.* Alfred A. Knopf, New York, 2014. $30.00. 432 pp., hardcover. ISBN 978-0-307-59980-3.

"What is reality?" "The proton mass is 938.3 MeV." *Our Mathematical Universe* is a remarkable work in which practical facts and philosophical questions exist (perhaps even happily) side by side.

Our Mathematical Universe is many things. For one, it is a lively and passionate telling of the personal story of a prominent cosmologist. Given the astonishing transformation the field of cosmology (the study of the history of the cosmos) has seen during Tegmark's career, his significant contributions to these developments, and his energetic and open way of writing (with a fun dose of the earlier history of the field thrown in), this book is highly worthwhile and will be loved by many. But Tegmark seeks to accomplish much more. A great deal of this work is devoted to exploring the nature of physics itself, and especially its relationship to mathematics. In this sphere, Tegmark transcends mere exploration and engages in the strong advocacy of a particular set of views.

In many circles, physics is seen as the most rigorous science. While some fields slip quickly into hand-waving arguments, physics has "fundamental equations," with "fundamental constants," and many of the fundamental insights they represent appear to form the bedrock of the rest of science, engineering, and life itself. When other fields of research appear less rigorous, this often can be due to the enormous difficulty inherent in computing the behavior of the large complex systems that describe molecules, materials, living things, and other parts of our world based on the quantitative description of their component parts in terms of fundamental physics.

However, some areas of physics are devoted to looking beyond the established fundamental equations in the hopes of finding something better: more fundamental, more beautiful, or perhaps more complete. Sometimes this search is driven simply by curiosity, but, as is often the case in the field of cosmology, the tension that arises when our fundamental picture seems inadequate to address the questions we care about can drive the search for an enhanced understanding and reveal a very different face of physics. While other fields of science that emerge in one way or another from the underlying physics can at least be guided by their emergence, the questions of what steps and vision might guide us to a deeper understanding of physics itself is extremely subjective. Ideas of what forms a good path forward are diverse, and sometimes the field can even be split about what constitutes progress. When this aspect of physics is viewed up close, our reputation for solidity and rigor can seem difficult to reconcile with the chaotic processes by which we search for a deeper understanding.

Still, our reputation for rigor does not come from nowhere, and this reputation is highly valued by even the most adventurous explorer on the path to deeper insights.

Publishers are invited to send books for review to Book Reviews Editor, SIAM, 3600 *Market St.,* 6^{th} *Floor, Philadelphia, PA* 19104-2688.

Our goal, after all, is to ultimately find laws of physics that are even more solid and rigorous than those we have discovered so far. Most who venture into these uncharted areas of physics reconcile the chaotic nature of the process with the wish for rigor by setting boundaries of one sort or another. Of course, there is often disagreement about where these boundaries should lie. Some feel that quantum mechanics must remain part of any deeper picture, while space and time itself could well become something that only emerges from a very different fundamental picture (perhaps loops of string) when the physical world is viewed from a limited everyday perspective. Others take the opposite view, holding onto spacetime "all the way down," but readily giving up quantum mechanics for a vision of a deeper theory. Then, generally, everyone has some particular tolerance of how wild or poorly-formed they will let the conversation become before stepping away from it, declaring the conversation too ill-defined or unlikely to get anywhere. The debate about which boundaries are appropriate is itself often passionate and can add to the impression of chaos at the frontier. Still, these boundaries do often have a positive role in keeping research moving in a positive direction.

Among all the colleagues I have known in my career, two stand out as being least limited by such boundaries: John Wheeler (who Tegmark and I and many others regard as a heroic figure) and Max Tegmark. Wheeler (inventor of the term "black hole" and known for his major contributions to gravitational and nuclear physics) was not uniformly this way, and the story of his discomfort in the 1950s with his student Hugh Everett's (then) revolutionary ideas about quantum physics is a fascinating part of the history of physics. (The story of Wheeler and Everett is not specifically discussed in this book, although it is hinted at, but it is covered in some of the recommended reading.) By the time I started my first postdoctoral position, with an office across the hall from Wheeler's (at the University of Texas at Austin), he was expressing bold and difficult-to-pin-down ideas about the connections between physics and information. I admit I was a bit wary of falling under his spell lest I became so distracted by vaguely posed ideas such as "it from bit" that I would never write another physics paper of any substance. But by the time I met Tegmark I had relaxed my boundaries considerably, and the numerous adventurous discussions we have had about all aspects of physics have been a very special part of my research life.

Generally, to a given physicist, colleagues with boundaries much more open than their own seem at least a little bit lost and unproductive, while colleagues with much tighter boundaries seem unreasonably conservative and narrow. An interesting thread that runs throughout this book tracks Tegmark's path as he navigates his own boundaries and those of his colleagues. An email quoted verbatim from an (unnamed) senior colleague and journal editor warns of dire consequences for Tegmark's future if he does not separate his "crackpot" ideas from his "serious research" (and perhaps abandon the former entirely). Tegmark also reflects on the experiences of others (including Everett and Einstein) whose best work pushed beyond the comfort zones of most of their colleagues. Happily, things have worked out just fine for Tegmark, and his work on both "serious" and "crackpot" projects has continued to thrive. To be fair, there are many others who perhaps should have more carefully heeded such warnings, and it is not yet clear whether Tegmark's work that so concerned the editor will eventually be understood as a significant contribution.

The ultimate focus of this book is Tegmark's "Mathematical Universe Hypothesis" (MUH) and the "Level IV Multiverse" that he argues follows from the MUH. The MUH itself is simply stated as follows: "Our external physical reality is a mathematical structure." (For Tegmark, using the word "is" as opposed to "is represented by" in this definition is crucial.) From there Tegmark argues for a kind of "mathematical democracy," where any mathematical structure is just as real as any other (even if we only experience one of them as "our reality").

Tegmark takes many gradual steps to reach these grand ideas. He carefully illustrates the different ways mathematics has often led the way in the development of physics, from the Everett many worlds, to the Friedmann cosmology, to inflationary cosmology (and the idea of "eternal inflation") and more recently the string landscape. He takes his time to explain each of these developments (and many others) in a relaxed and accessible manner. He works earnestly to define his terms, including "reality" and "mathematical structure" (and, of course, defines the Level I, II, and III Multiverses). In making his larger leaps, Tegmark takes to heart the words of Nobel Prize–winning physicist Steven Weinberg, who noted that often in physics "our mistake is not that we take our theories too seriously, but that we do not take them seriously enough."

To illustrate, from the point of view of the Level IV Multiverse there are presumably all possible realities: Some with spacetime "all the way down" but with emergent quantum physics, and some with the reverse. Some where neither spacetime nor quantum physics is fundamental, and others where both are. There is no absolute right answer as to which of these is true: They all are in the multiverse, although we still can ask which of them is true for "our particular reality."

Although mathematics occupies a central role in Tegmark's story, his narrative takes the informal style of a particularly easy-going physicist. For example, the three dimensions of our space are presented by asking, "how many pencils can you arrange so they are all perpendicular to each other?" (Presumably these pencils have bits of shavings clinging to them from the sharpener and well-chewed erasers.) The book is thoughtfully and clearly organized, with the thirteen chapters grouped into three large sections covering cosmology, elementary particles (especially quantum physics), and the Level IV Multiverse. Each chapter ends with a conveniently presented outline of the main points covered within, and the preface offers a diagram suggesting how readers with different backgrounds might approach reading the book.

Much of the first two sections of the book covers established physics (though often carefully designed to ultimately drive home more bold and controversial points). The reader will learn the basic distances and sizes corresponding to planets, galaxies, and so on, culminating in the question, "what is space?" Classic topics such as the cosmic microwave background and galaxy clustering and basic facts from atomic, nuclear, and particle physics are presented. Even the established topics are treated in a very relaxed and personal manner, such as Tegmark's reflections on the experience of doing research (including "the joys of being scooped" and that sign mistake that almost ruined a talk). Also, Tegmark shares some lovely personal impressions of some of our colleagues. Little attempt is made to provide a textbook-level systematic treatment, which I imagine will make this book more accessible and fun for many readers (an extensive "further reading" list is also presented, with more emphasis on comprehensiveness than on guiding the reader to the next few books to read on these subjects).

As the narrative moves toward the main, and more philosophical, points, the reading becomes a more challenging experience. I think any reader will, as I did, find many questions that are not addressed or at least not addressed in a satisfactory manner; this is perhaps not surprising for topics like "what is time?" and "what is consciousness?" But Tegmark just keeps on making the case for his passionately held vision. As I finish writing this review while at a conference in Germany, the latter parts of the book conjure up the image of being on a road trip with Tegmark, blasting down the autobahn at 150 mph, with Tegmark fully confident he knows where he is going, and that you will like it when you get there. Meanwhile, as the passenger, I keep wondering why he passed by this or that interesting place without stopping for a closer look, and how he is going to manage to avoid various obstacles that loom on the horizon.

One of my big questions has been how Tegmark's ways of thinking might change the way I go about my physics research, should I be brought on board. It is hard to see how anything would change. He suggests that if we find physical phenomena that do not allow a mathematical description, we will have falsified his MUH, but how should we know whether such a situation is just a temporary setback due to our own lack of creativity? In any case, I am still very curious about how physics works in "my particular reality," and still hopeful that my colleagues and I can extract insights from how physics has developed so far to guide us to a deeper understanding. I'm not sure how saying that this is just one of many possible realities will significantly change this. As Tegmark acknowledges, the idea of drawing our reality from an ensemble and making statistical predictions about what we will discover is not sufficiently well developed to yield clear results. Maybe if I were sold on Tegmark's approach I would invest some of my time in developing a technical understanding of the Level IV Multiverse and the search for ourselves within it, but here I run up against clashes with my own research (or boundaries, in the language I used above), in which I have come to regard probabilities and ensembles in a manner incompatible with Tegmark's program.

In the end, though, this book is the better for not convincing me (and no doubt many other readers) to jump on his bandwagon. As Tegmark readily admits, his ideas are not advanced enough for people to take sides. Instead, I have gained something much more. Reading this book has caused me to stop and reflect on my own boundaries and the ideas that create structure for my own research. Now I want to go back out on the autobahn in my own car, so I can stop at some of the places Tegmark whizzed by and look as long as I like. And perhaps I will end up on a different highway altogether. I believe every reader will finish this book with new questions in mind, questions that may well change how they approach their own work and how they see the world.

I should add that although I address this review to SIAM members who probably all have some kind of technical profession, I feel this book would be readily accessible to a much wider range of readers, in fact to anyone curious about how science works and where it is headed. For good measure, Tegmark throws in a cheerful dose of informality. He pauses here and there for entertaining notions such as the dominance of napkins (instead of envelopes) used for performing "back of the envelope calculations" or to offer a list of cool questions whose answer is "42." Also, his frank and warmhearted account of the ups and downs of starting a career in physics will resonate with and offer support to individuals considering or starting out on such a path themselves.

The relationship between mathematics and the physical world is a truly wonderful thing. It is something most definitely worthy of a pause to celebrate and reflect upon. And those of us whose work exploits this relationship can especially hope that insights into this relationship can guide and enhance our work. Tegmark is to be congratulated for boldly and very accessibly sharing his radical and very thoughtful reflections with all of us. The reader is treated to a clear, personal, and passionate account of exciting advances in physics, especially in cosmology, over the last several decades, and of the joys of being part of these developments. The reader is also exposed to bold and controversial ideas about the relationships between physics, mathematics, and reality. The subject matter is way too subjective, and the book leaves too many questions unanswered, to expect that a majority of readers will become convinced of all of Tegmark's bold positions, but I expect most readers will get a great deal out of these adventures, as I did, because of the way he stimulated and challenged my own thinking on these deep topics. While I feel lucky that I don't need to think about all of this to write interesting physics papers, I have emerged from reading this book more aware and more curious about how my beliefs about these deeper questions shape my research and my perceptions about the world around me.

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Stability of Functional Equations in Random Normed Spaces. *By Yeol Je Cho, Themistocles M. Rassias, and Reza Saadati.* Springer, New York, 2013. $109.00. xx+246 pp., hardcover. ISBN 978-1-4614-8476-9.

Several books on stability of functional equations have been published in the last few years, but this one by Yeol Je Cho, Themistocles M. Rassias, and Reza Saadati is quite exceptional, because it is the first to consider the issue of such stability in random normed spaces. Moreover, it focuses only on random spaces.

The book is Volume 86 of the series *Springer Optimization and its Applications*, but there are actually no comments in the book on connections between the stability of functional equations and the optimization issues. Certainly, some connections exist and experienced mathematicians notice them easily, but for those less experienced it can be quite difficult. The preface, which introduces the main subjects of the book, gives only a brief account of the history of functional equations and their stability, with some references to suitable monographs and surveys.

At present, it is commonly held that investigations of the stability of functional equations began with a question raised by S.M. Ulam in 1940 and a partial answer to it published by D.H. Hyers [2] in 1941. We quote that question below, but we are not actually sure what motivated Ulam to pose it, though some indications can be deduced from the remarks in [4]. For instance, maybe the motivation was the observation that a natural phenomenon is often subject to disturbances, which means that it should be described by inequalities rather than by equations. Therefore, it is important to know when, why, and to what extent we can replace those inequalities by suitable (or corresponding) equations.

The question of Ulam has been quoted in numerous papers on the subject, in various forms, but mainly as follows (cf. [2]):

Let G_1 be a group and (G_2, d) a metric group. Given $\epsilon > 0$, does there exist $\delta > 0$ such that if $f : G_1 \to G_2$ satisfies

$$d(f(xy), f(x)f(y)) < \delta, \qquad x, y \in G_1,$$

then a homomorphism $T : G_1 \to G_2$ exists with

$$d(f(x), T(x)) < \epsilon, \qquad x \in G_1?$$

Below we present the answer of Hyers [2].

Let X and Y be Banach spaces and $\varepsilon > 0$. Then for every $g : X \to Y$ with

(1)
$$\|g(x+y) - g(x) - g(y)\| \le \epsilon, \qquad x, y \in X,$$

there exists a unique function $f : X \to Y$ such that

$$\|g(x) - f(x)\| \le \epsilon, \qquad x \in X,$$

and

$$(2)\quad f(x+y) = f(x) + f(y), \qquad x, y \in X.$$

The latter result says that *the Cauchy functional equation* (2) *is Hyers–Ulam stable (or has the Hyers–Ulam stability) in the class of functions mapping X into Y* (i.e., in Y^X).

Let us now mention that we are aware of an earlier result of this type, due to