Physics 262 *Early Universe Cosmology* (A. Albrecht)
Homework 6
Assigned Wednesday May 19
Due Wed June 9 11pm (on canvas)
This HW carries 40 points.

*You have ~3 weeks, but this is a LONG HW and the deadline is a **hard deadline** this time. Arsalan and I will be on a very tight schedule to finish grading and submit your final grades in time. Please allow yourself plenty of time to work on this.*

6.1) Consider your standard cosmological model from HW1/2. Evaluate the possible values of

$$\frac{\rho_k\left(T=10^{16}GeV\right)}{\rho_c\left(T=10^{16}GeV\right)} \tag{1.1}$$

that are consistent with $|\rho_k| < 0.1\rho_c$ today. For this purpose you may assume g_* is a constant throughout the history of the universe. *(Note, this last assumption is not a great approximation, but it does not interfere with the point of this problem.)*

6.2) Consider a comoving length that, when the temperature was $T=10^{16}GeV$ was equal to the horizon distance at that time. What is the ratio of that comoving distance today to the Hubble length today? *Use the standard cosmology from HW1/2 and take g_* to be constant.*

6.3) Write a program to integrate K&T's equation 8.14. Assume ρ_φ (from eq 8.20, neglecting gradients) is the only contribution to H and use $V(\varphi)=\frac{1}{2}m^2\varphi^2$. Solve for $\log(a(t))$ (which is easier to solve for than $a(t)$) but allow the initial value $a(t_i)$ to be arbitrary. For $m=10^{16}GeV$ (the grand unification scale) find a suitable initial value of φ and time range to see $\log(a(t))$ change by at least 100 during an inflationary period (when the energy density is potential dominated) and then follow at least two oscillations of φ following the end of the inflationary period. Make separate plots of $\varphi(t)$, $\log(a(t))$, $w(t)$, $\left.\frac{\delta\rho}{\rho}\right|_H \equiv \delta_H = \left(\frac{HV'}{2\pi\dot{\varphi}^2}\right)$ (from K&T Eqn 8.51) and $\rho_\varphi(t)$ for your solution. In your plot of $\rho(t)$ convert $\rho(t)$ to the same units you used for $\rho(a)$ calculated in earlier homeworks. How do your values of δ_H during inflation ($w \approx -1$) compare with the realistic value $\delta_H \approx 10^{-5}$?

Hints:

- *Give some thought to units. I found working with everything in GeV (including $G=m_p^{-2}$) was good, but you may have your own preference.*

- *Plot w(t) (use K&T equations 8.20 and 8.21 and recall that we are setting gradients to zero) and use trial and error to find a suitable initial conditions and time range*
- *You can choose $\dot{\varphi}(t_i) = 0$.*
- *You should be able to do this while keeping the integration time relatively short if needed. I chose $t \in (0, 100m^{-1})$ for fast turnaround, although your program probably can handle longer integration times without any trouble.*
- *Explore solution space by experimenting with initial values of φ rather than by lengthening the integration time.*
- *Use equations 8.20 and 8.21 to work out w(t) (remember we are neglecting gradients).*
- *The assumption that ρ_φ dominates H is justified after an initial period of inflation drives the universe to critical density (I don't want you to model that part however). .*

6.4) The density perturbations generated during inflation can be characterized by the power spectrum $P(k) = <\delta_k \delta_{k'}>$ where δ_k is the Fourier transform of the density perturbation $\delta\rho/\rho$. Inflation predicts a "scale-free" power spectrum given by $P(k) \propto k^{n_s - 1}$, where n_s is called the spectral index. We want to find this quantity from your answer to 6.3).

i) In your solution to 6.3, check that far from the minimum, $\dot{\phi}^2 \ll V(\phi)$. Define the "slow-roll" parameter $\epsilon = -\frac{\dot{H}}{H^2} \approx \frac{1}{16\pi G}\left(\frac{V'(\phi)}{V}\right)^2$ (prime denotes derivative wrt φ). Show that, in order to get *accelerated* expansion ($\ddot{a} > 0$), we must have $\epsilon < 1$.
Hint: You can do this a few ways. One of them is to use the second Friedmann eqn

ii) Define another slow-roll parameter, $\eta = \frac{1}{8\pi G}\left(\frac{V''(\phi)}{V(\phi)}\right)$. Find both η and ϵ in the slow-roll regime (i.e. far from the minimum). Intuitively, this means that the potential must be flat ($\epsilon \ll 1$) and stay flat for long enough ($\eta \ll 1$) for the Universe to inflate sizably.
Hint: The exact point where you find the quantities is not so important since both remain roughly constant in the slow-roll regime.

iii) The spectral index is given by $n_s = 1 + 2\eta - 6\epsilon$. Find n_s and compare your result to the Planck constraint given in Table 2 of https://arxiv.org/pdf/1807.06209.pdf (pick any column).
Note: the significant difference from the Planck value may seem alarming but that's a result of the particularly simple inflationary potential being considered here. The important point is that inflation generically gives a slight "red tilt" ($n_s < 1$). Conversely, the Planck measurement shows how observations can help us distinguish between various inflationary models.

6.5)

i) Repeat the procedure in problem 6.3 but this time vary *m* until you find a case where $\delta_H \approx 10^{-5}$ (within a factor of 3 or 4).

ii) Since P(k) goes as the *square* of this quantity, we can insert a proportionality constant in the expression for P(k) in 6.4 to write, $P(k) = A_s k^{n_s - 1}$. From K&T Eqn. 8.53, you can get that $A_s \approx 2\pi^2 \delta_H^2$. Compare your value of A_s to the Planck value in the same table as before.

6.6)

i) Inspect your solution in problem 6.3 and identify a "reheating time" t_r when *w* first starts to vary significantly from -1. Just make a choice from your plot. This does not need to be high-precision.

ii) Find $\rho_\varphi(t_r)$

iii) Consider the cosmological model from HW2. Suppose *g** remains constant for all $a < 10^{-6}$ (unrealistic, but good enough for our purposes). Find the value of the scale factor in this model where the total energy density equals $\rho_\varphi(t_r)$ from part ii). In what follows, we'll call this value a_r *Hint: You do not need to integrate for scale factors* $a < 10^{-6}$ *to answer this. Just use your solution from HW 2. and extrapolate in a simple way.*

iv) What is the temperature $T(a_r)$ in your model in part iii?

v) Consider the co-moving length scales with values 1Mpc, 100Mpc and 1000Mpc today. What values do these co-moving lengths when $a = a_r$? What are the values of the scale factors when each of these co-moving lengths "re-enters" the Hubble length.

You now can consider the following simplified cosmological model that includes inflation: for $a > a_r$ the cosmology is given by your model in problem 6.6, part iii. For $a < a_r$ the cosmology is given by your solution in problem 6.3. To get the scale factor right all you have to do is re-scale *a(t)* in your solution to 6.6iii) so that $a(t_r) = a_r$. This does not change the validity of the solution since the overall scale of *a* is just a matter of convention. Also, you need not worry about matching values of *t* at a_r. There is no need to define a global time coordinate. This cosmological model assumes instant "reheating": When $a = a_r$ the matter of the universe converts instantly from pure scalar field matter to the various matter components in your model form HW 2. (The reheating is not really "instant", but that assumption allows you to build a cosmological model that is pretty good for pedagogical purposes.)

6.7) Make a table of values of $\left.\frac{\delta\rho}{\rho}\right|_H$ for each of the co-moving lengths given in problem 6.6 part v (evaluated at the re-entry points determined in 6.6 part v). *Hint: To do this you will need to extrapolate back to the Hubble length exit time for each of these co-moving* *lengths. I expect this will be especially straightforward to work out graphically by plotting various relevant properties from above and finding Hubble exit by eye.*

6.8) Produce a table with three columns giving photon temperature in GeV, photon temperature in Kelvin, and time (in the time coordinate from the model from HW2) in units of seconds. The table should contain entries for the following events in your cosmological model:

- reheating,
- nucleosynthesis *(just use T=1MeV)*
- radiation-matter equality
- last scattering *(scale factor* $= \frac{1}{1100}$*)*
- the three Hubble length re-entry points you produced for problem 6.6 v).
- today

Hint: I think graphical methods similar to those suggested above might be useful here too.