Physics 262 Early Universe Cosmology
Homework 5
Assigned May 6
Due May 27 11pm (uploaded on canvas)

These papers may be helpful:
https://arxiv.org/abs/astro-ph/9711102 (especially for problem 5.1)
https://arxiv.org/abs/astro-ph/9908085 (for problem 5.2)
However, it is possible to do fine without reading these papers.

NOTE: This homework typically takes some playing around to allow you to find the requested solutions. Please allow time for that and be patient with the process. Also, we can discuss how things are going in class. If you bring in code that is giving puzzling results we can project it and talk through how to get there.

5.1) Consider a homogeneous scalar field evolving according to K&T Eqn. (8.14), with $V(\varphi) = V_0 e^{-\lambda\varphi}$. *You also will need K&T Eqn (8.20) and Eqn (8,21) for what follows.*

a) Show analytically that if the only components of the Universe are non-relativistic matter and a homogeneous scalar field φ (and assuming $\rho_k = 0$), a solution exists where ρ_φ remains a fixed fraction of ρ_m and $V(\varphi) = \frac{1}{2}\rho_\phi(\varphi)$. *Hint: You probably want to just do this by substitution.*

b) Give an expression for $\frac{\rho_\varphi}{\rho_{tot}}$ in terms of λ.

c) For what values of λ does your answer to b) make sense?

d) Verify that the "equation of state parameter" $\frac{p_\varphi}{\rho_\varphi}$ has the value it should for this solution.

One model of dark energy has a homogeneous scalar field obeying K&T Eqn. (8.14) with

$$V(\varphi) = V_0\left(\chi(\varphi - B)^2 + \delta\right)e^{-\lambda\varphi} \tag{1.1}$$

The next few problems will deal with this case.

You should incorporate what we discuss about this model in class into your approach to problem 5.2

5.2) Consider a simple two component model where made up of only ρ_m and ρ_φ, in the case where

$$
\begin{aligned}
\lambda &= 8 \\
V_0 &= 1 \\
\delta &= 0.005 \\
\chi &= 1 \\
\rho_r &= \rho_k = 0
\end{aligned}
\tag{1.2}
$$

Here I use "reduced Planck units" where $8\pi G \equiv 1$. Solve K&T Eqn. (8.14) and experiment with a variety of initial values of φ. *For each case I recommend that you choose an initial value for ρ_m that obeys the scaling solution you found in problem 5.1.. This recommendation is just to offer you a starting point, and you will probably want to fiddle around with it to get a solution without too many transients.* To hand in:

a) For this part, let $B = 34$. On the same graph, plot $V(\varphi)$ given by Eqn 1.2 in and $V(\varphi) = V_0 e^{-\lambda\varphi}$ for the parameters given in Eqn. (1.2). Chose a range for φ that includes $\varphi = B$ and extends far enough to either side of B that the two curves become similar (away from $\varphi = B$). Zoom in to make sure you are able to see a local minimum around $\varphi = B$ (Important: See hint iii at the end to understand what I mean by "similar" here.)

b) In reality, we don't need to assume a value of B. Let $H_0 = 67 km/s/Mpc$ and $\omega_m^0 = 0.14$

   i. Find ρ_φ^0 (i.e. ρ_φ today)

   ii. Find (possibly numerically) the location of the minimum of the potential, $\varphi_{\min}$, in terms of B.

   Hint: If your numerical solver can not find the minimum for an unknown B, leave it as a function of B and pass it to the following part

   iii. Now suppose that the field is at the minimum of the potential and has zero velocity today (i.e. $\varphi(t_0) \equiv \varphi_0 = \varphi_{\min}$ and $\dot{\varphi}(t_0) \equiv \dot{\varphi}_0 = 0$). Use your answers to (i) and (ii) to solve for B. It's just numerically inverting the equation $V(\varphi_{\min}(B)) = \rho_\varphi^0$

   Hint: You should find a value of B not very different from the value we assumed in (a)

c) Find a numerical solution for $\varphi(t)$ by solving K&T Eqn. (8.14). Note that now $H^2 = \rho_m + \rho_\varphi$

   Hint: In order to solve the (2nd order) differential equation you will need initial conditions for $\dot{\varphi}_{ini}$ and φ_{ini} at some initial time a_{ini}. Use the fact that initially the field is well-approximated by the pure-exponential potential so that you know what $w(\varphi)$ and ρ_φ ought to be. Given a choice for a_{ini} you can then solve for the initial conditions of the field.

d) Plot Ω_φ and Ω_m as a function of a or t (whatever is convenient) for the solutions in your answer to 5.2b) and 5.2c).

e) Make a single two panel plot showing the solutions you found in problem 5.2c. In the top panel plot φ on the x-axis and *t* or *a* on the y-axis. In the lower panel plot $V(\phi)$. Make sure the x-axis is the same on both panels. *This plot will help you see where the field is moving in the potential as a function of time*

f) One problem that has recently captivated cosmologists is the so-called "H_0 Tension" wherein the values of H_0 inferred from two very different measurements differs by $\sim 4.5\sigma$ (!!). A potential resolution might be to utilize a quintessence field of the sort studied here. Re-do part (b), but this time assume $H_0 = 72 km/s/Mpc$. What is the new value of B and the corresponding value of ρ_φ^0?

<u>Hints</u>

i) *To do the numerical integration I recommend Matlab function "ode45". (If you are using Mathematica, Arsalan, who is doing something similar for his research project can tell you about his good and bad experiences with different solvers for this sort of integration.)*

ii) *You will need to integrate simultaneously to get $\rho_m(t)$. One way to do this is to solve for $a(t)$ and use $\rho_m \propto \frac{1}{a^3}$. But it probably does not make sense to use the $a_0 = 1$ convention here. In particular, this homework is a theoretical exploration rather than a realistic model of the cosmos. I do not expect you to be concerned with which if any parts of your calculations might correspond to "today" ($a = a_0$).*

iii) *Here is an additional hint for problem 5.2a. We can use:*

$$f(\varphi) \equiv \left(\chi(\varphi - B)^2 + \delta\right) \tag{1.3}$$

to write

$$V(\varphi) = V_0 f(\varphi) e^{-\lambda\varphi} = V_0 \exp\left(-\lambda\left(\varphi + \frac{\ln\left(f(\varphi)\right)}{\lambda}\right)\right) e^{-\lambda\varphi}. \tag{1.4}$$

When the logarithmic term is sufficiently slowly varying (away from $\varphi = B$) than Eqn. (1.4) approximates the form $V(\varphi) = V_0 e^{-\lambda\varphi}$ but with a constant offset for φ. Under those conditions the two expressions are similar in their behavior (exponential), but they will not look the same when plotted together due to the offset that comes in from the prefactor.