The Friedmann Robertson Walker Cosmological Model

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The Friedmann-Robertson-Walker (FRW) cosmological model is arrived at by applying an extreme set of symmetries ("homogeneity and isotropy") to the Einstein Equations. As shown on pp 44-47 on Ryden, some of these expressions seem to be derivable from Newtonian arguments, but the only careful and complete treatment requires General Relativity.

1. Parameters and Degrees of Freedom

SPACETIME

The symmetries reduce the number of degrees of gravitational degrees of freedom from infinity (many dynamical degrees of freedom at each point on a continuous 3D space) to

- one dynamical degree of freedom $a(t)$ and
- two constants, the curvature $\frac{\kappa}{R_0^2} \equiv k$ and Λ.

The metric is given by

$$ds^2 = dt^2 - a(t)^2 \left(\frac{dr^2}{1-kr^2} + r^2 d\Omega \right) \tag{1.1}$$

Where r is a radial "commoving coordinate" and $d\Omega$ is some angular part which can often be taken to be zero for certain calculation without loss of generality. Also note that the Ω in Eqn.(1.1) is different from that which appears to represent the reduced densities below.

MATTER

The degrees of freedom describing matter that appear in Einstein's equations are also greatly reduced. They are

- $\rho(t)$ (the uniform energy density in the universe) and
- $p(t)$ (the uniform "pressure" in the universe)

Each of these are often written a sum of individual components, which helps one understand their evolution, but it is only the total value of each of these that appears in Einstein's equations. Also, $p(t)$ is a generalization of the pressure that need not have all the familiar properties (it can be negative for example).

2. Equations of Motion

The equations of motion for $a(t)$ (from Einstein's equations) is given by the following. The 1st Friedman equation is

$$\left(\frac{\dot{a}}{a}\right)^2 \equiv H^2 = \frac{8\pi}{3m_P^2}\rho - \frac{k}{a^2} + \frac{\Lambda}{3} \tag{2.1}$$

or

$$\left(\frac{\dot{a}}{a}\right)^2 \equiv H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3} \tag{2.2}$$

or

$$\left(\frac{\dot{a}}{a}\right)^2 \equiv H^2 = \frac{8\pi}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3} \tag{2.3}$$

where the difference versions just depend on unit conventions. All versions use

$$c \equiv 1 \tag{2.4}$$

Equation (2.1) uses energy units with

$$m_P^2 \equiv \frac{1}{G} \tag{2.5}$$

And equation (2.3) uses geometrized units for which

$$G \equiv 1 \tag{2.6}$$

The 2nd Friedmann Equation is

$$\left(\frac{\ddot{a}}{a}\right) = -\frac{4\pi}{3m_P^2}\left(\rho + 3p\right) + \frac{\Lambda}{3} \tag{2.7}$$

Only the first Friedman Equation is required to solve for $a(t)$ except when $\dot{a} = 0$.

The local conservation of energy gives the "fluid equation" for ρ and p:

$$\dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho + p\right) \tag{2.8}$$

which can also be written (using Eqn (4.1)):

$$\frac{d\rho}{\rho} = -3(1+w)\frac{da}{a} \tag{2.9}$$

or as

$$d\left(\rho a^3\right) = -pd\left(a^3\right) \tag{2.10}$$

(K&T Eqn. 3.5)

Any two of the equations (2.1), (2.7) and (2.8) can by used to derive the other one. This degeneracy is not true for the full Einstein and continuity equations. It only comes about from the massive reduction in the number of degrees of freedom due to the assumption of homogeneity and isotropy.

3. Converting between different conventions

Ryden

Ryden's version of Eq. (2.1) is

$$\left(\frac{\dot{a}}{a}\right)^2 \equiv H^2 = \frac{8\pi G}{3}\frac{\varepsilon}{c^2} - \frac{\kappa c^2}{R_0^2 a^2} + \frac{\Lambda}{3} \tag{3.1}$$

(this is equation 4.62 from Ryden[1], which first appears as 4.13 with $\Lambda = 0$).
Her version of Eq (2.7) is

$$\left(\frac{\ddot{a}}{a}\right) = -\frac{4\pi G}{3c^2}\left(\varepsilon + 3P\right) + \frac{\Lambda}{3} \tag{3.2}$$

(Ryden equation 4.64, which first appears as 4.44 with $\Lambda = 0$)
The Ryden version of the local conservation of energy (Eqn (2.8)) is:

$$\dot{\varepsilon} = -3\frac{\dot{a}}{a}\left(\varepsilon + P\right) \tag{3.3}$$

which can also be written:

$$\frac{d\varepsilon}{\varepsilon} = -3(1+w)\frac{da}{a} \tag{3.4}$$

(Ryden equation 5.8) Ryden's ε are related by ρ

$$\varepsilon = \rho c^2 \tag{3.5}$$

So they are equal in units where $c = 1$.

Kolb & Turner

K&T's version of Eq. (2.1) is

[1] As discussed on p 43, Ryden chooses a popular convention where the curvature constant as $k = \frac{\kappa}{R_0^2}$ where $\kappa = \pm 1, 0$ and R_0 is positive.

$$\left(\frac{\dot{R}}{R}\right)^2 \equiv H^2 = \frac{8\pi G}{3}\rho - \frac{k}{R^2} + \frac{\Lambda}{3} \tag{3.6}$$

With $\Lambda = 0$ this is equation 3.10 from K&T (slightly rearranged)
K&T's version of Eqn. (2.7) is

$$\left(\frac{\ddot{R}}{R}\right)^2 = -\frac{4\pi G}{3}\left(\rho + 3p\right) + \frac{\Lambda}{3} \tag{3.7}$$

with this is K&T's equation 3.11[2].
The K&T version of the local conservation of energy (Eqn. (2.8)) is:

$$\dot{\rho} = -3\frac{\dot{R}}{R}\left(\rho + p\right) \tag{3.8}$$

This can also be written:

$$\frac{d\rho}{\rho} = -3(1+w)\frac{dR}{R} \tag{3.9}$$

These appear as K&T's equations 3.5 and 3.6 which are in slightly different forms.

NB: K&T absorb a factor of R_0 into the scale factor:

$$R \equiv aR_0 \tag{3.10}$$

which gives it units of length. In the case where $k = 0$, $R_0 = \infty$. In this case they give R_0 an arbitrary finite value in Eqn. (3.10) without loss of generality.

4. *Equation of state*

Often one can relate ρ and p through a simple "equation of state":

$$p = w\rho \tag{4.1}$$

in which case one can solve Eq. (2.9) for $\rho(a)$ and then use that to solve Eq (2.1) for $a(t)$. If w is constant, one gets

$$\rho_w(a) = \rho_{w,0}a^{-3(1+w)} \tag{4.2}$$

(Similar to Ryden Eq 5.9)

Useful choices of w are

- $w = 0$ "non relativistic matter", "dust", "pressureless matter", "matter" (often written as ρ_m)
- $w = 1/3$ "relativistic matter", "radiation" (often written as ρ_r)

[2] If you see the Friedman equations with $\Lambda \neq 0$ somewhere in K&T, please tell me.

5. Useful definitions

The critical density:

$$\rho_c \equiv \frac{3c^2}{8\pi G} H^2 \tag{4.3}$$

(Ryden Eq 4.25)
and

$$\Omega \equiv \frac{\rho}{\rho_c} \tag{4.4}$$

(Ryden Eq 4.28)

Sometimes the energy density is written as a sum of different components (such as matter and radiation):

$$\rho = \sum_i \rho_i \tag{4.5}$$

in which case one often defines

$$\Omega_i \equiv \frac{\rho_i}{\rho_c} \tag{4.6}$$

Furthermore, often each ρ_i and corresponding pressure p_i independently obey Eq. (4.1) and Eq. (2.9) to give

$$p_i = w_i \rho_i \tag{4.7}$$

which can be used in Eq. (2.9) to separately determine each $\rho_i(a)$.

Sometimes the 2nd and 3rd terms of Eq. (2.1) are replaced with equivalent "effective energy densities"

- $\frac{8\pi G}{3}\frac{\rho_\Lambda}{c^2} \equiv \frac{\Lambda}{3}$ (Λ-matter with $w = -1$)
- $\frac{8\pi G}{3}\frac{\rho_k}{c^2} \equiv -\frac{k}{a^2}$ k-matter (curvature matter) with $w = -1/3$ (Note ρ_k is negative for positively curved universes ($\kappa = +1$).

Using this notation,

$$\rho_c \equiv \frac{3}{8\pi G} H^2 = \sum_i \rho_i + \rho_k + \rho_\Lambda \tag{4.8}$$

where the index *i* runs over the different matter-energy components and that last two terms represent the *k* and Λ terms in the Friedmann equation. Dividing Eqn (4.8) through by ρ_c gives

$$1 = \sum_i \Omega_i + \Omega_k + \Omega_\Lambda \tag{4.9}$$

Note also that by definition,

$$\rho_{Tot} \equiv \rho_c - \rho_k \tag{4.10}$$

Remember, ρ_{Tot} is defined in this way to facilitate discussions of curvature. So that $\Omega_{Tot} = 1 \Rightarrow \Omega_k = 0$

6. *Distances etc*

Many of these topics are treated in chapter 7 of Ryden.

Proper Distance

The proper distance is the physical distance between two locations at a fixed cosmological time. If one chooses one location at the zero of radial commoving coordinate r

$$d_p(t) \equiv a(t)\int_0^r dr = a(t)r \tag{5.1}$$

Light travel distance

For photons $ds^2 = dt^2 - a^2dr^2 = 0$ so

$$dt = adr \tag{5.2}$$

or

$$r = \int \frac{dt}{a} \tag{5.3}$$

The physical distance traveled by a photon between times t_1 and t_2 is

$$a(t_2)r_{12} = a(t_2)\int_{t_1}^{t_2} \frac{dt}{a}. \tag{5.4}$$

Horizon

The horizon distance is typically defined as the distance light has traveled since the initial singularity. At any time t_2 this is given by

$$d_H \equiv a(t_2)\int_0^{t_2} \frac{dt}{a(t)} = a_2 \int_0^{a_2} \frac{da}{a^2 H}. \tag{5.5}$$

Conformal Time

The conformal time η is defined by

$$dt = ad\eta. \tag{5.6}$$

The invariant line element can then written

$$ds^2 = a^2\left(d\eta^2 - dr^2\right) \tag{5.7}$$

Using the usual convention $\eta(a=0)=0$ gives

$$\eta\left(t_2\right) = \int_0^{t_2} \frac{dt}{a\left(t\right)} = \int_0^{a_2} \frac{da}{a^2 H} \tag{5.8}$$

so

$$d_H\left(t_2\right) = a_2\eta_2 \tag{5.9}$$

Luminosity Distance

In flat space the flux (power/area) at a distance d from an object with luminosity L is given by

$$f = \frac{L}{4\pi d^2} \tag{5.10}$$

so one can use flux measurements of objects of know luminosity to infer the distance. In an FRW universe the flux depends on the curvature and evolution of the universe, and Eqn (5.10) generalizes to

$$f = \frac{L}{4\pi d_L^2} \tag{5.11}$$

where

$$d_L(a_i) = \frac{1}{a_i}\begin{cases} \frac{1}{\sqrt{|k|}}\sinh\left(\sqrt{|k|}\chi\left(a_i\right)\right) & k<0 \\ \chi\left(a_i\right) & k=0 \\ \frac{1}{\sqrt{|k|}}\sin\left(\sqrt{|k|}\chi\left(a_i\right)\right) & k>0 \end{cases} \tag{5.12}$$

The index "*i*" refers to the value when the light is emitted and

$$\chi\left(a_i\right) = \eta_0 - \eta_i \equiv \int_{a_i}^{1} \frac{da}{a^2 H\left(a\right)} = H_0^{-1}\int_{a_i}^{1} \frac{da}{a^2 \frac{H\left(a\right)}{H_0}} \tag{5.13}$$

with $\sqrt{|k|} \equiv H_0\sqrt{|\Omega_k|} = \frac{H_0}{h}\sqrt{|\omega_k|}$. Typically one discusses the luminosity distance for objects observed today with $a_0 = 1$ as shown in Eqn (5.13), although in theory one can calculate luminosity distance between any two points. Equation (5.12) can be rewritten as

$$d_L(a_i) = \frac{H_0^{-1}}{a_i}\begin{cases} \frac{h}{\sqrt{|\omega_k|}}\sinh\left(\sqrt{|k|}\chi\left(a_i\right)\right) & k<0 \\ \frac{\chi\left(a_i\right)}{H_0^{-1}} & k=0 \\ \frac{h}{\sqrt{|\omega_k|}}\sin\left(\sqrt{|k|}\chi\left(a_i\right)\right) & k>0 \end{cases} \tag{5.14}$$

Which shows that H_0^{-1} sets the scale for d_L. This point may seem a little circular for the $k=0$ case, but inspection of Eqn (5.13) shows that pulling out a factor of H_0^{-1} is natural, because it leaves the quantity $\frac{H\left(a\right)}{H_0}$ inside the integral. That gives the a dependence with the overall scale factor out.

Angular Diameter Distance

In flat space on the angle subtended by an object of length l a distance d away is (using the small angle approximation)

$$\theta = \frac{l}{d} \tag{5.15}$$

In FRW space the observed angle depends on the curvature and evolution of the universe and Eqn (5.15) generalizes to

$$\theta = \frac{l}{d_A} \tag{5.16}$$

where the "angular diameter distance" d_A is given by

$$d_A = a_i^2 d_L \tag{5.17}$$

Redshift

If a photon is emitted at time t_1 with at wavelength λ_1 and propagates through an FRW universe until it is observed at t_2 the observed wavelength is

$$\lambda_2 = \frac{a_2}{a_1}\lambda_1 \tag{5.18}$$

For observations made today one defines the "redshift" z by

$$1+z \equiv \frac{a_0}{a_i} = \frac{1}{a_i} \tag{5.19}$$

so the observed wavelength λ_O is

$$\lambda_O = \left(1+z\right)\lambda_i\,. \tag{5.20}$$

Thus the shift of recognizable spectral lines can be related directly to a_i.

Note, just as the scale factor can be used as a measure of time (to the extent that it is monotonic in *t*), so the redshift can be used as a measure of time. Redshift often does show up as a measure of time in cosmological discussions (in is likely to do so in some of the slides I present in this class).

Note that for the above equations to hold the wavelengths must be measured in a frame locally at rest in the cosmological frame.

Magnitude Distance Modulus

The apparent magnitude of an object observed to have flux f is defined by

$$m \equiv -2.5\log_{10}\left(f/f_x\right) \tag{5.21}$$

where $f_x = 2.53\times10^{-8} watt\times m^{-2}$. The absolute magnitude M is defined as the apparent magnitude an object would have if it were placed at a distance of 10pc. Using the generalization of the inverse square law of radiation (Eqn (5.11)) gives

$$M = m - 5\log_{10}\left(\frac{d_L}{1Mpc}\right) - 25 \tag{5.22}$$

The "Distance Modulus" for an object at redshift z_i is given by

$$\mu\left(z_i\right) \equiv m - M = 5\log_{10}\left(d_L\left(z_i\right)\right) + 25 \tag{5.23}$$

Where d_L is measure in Mpc.