Inflationary Cosmology

Introduction in The Art of Universe Making

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Cudos

This text emerged from the lectures the author gave to advanced graduate students in cosmology and high energy physics at UC Davis in the period 2002-2005. Abridged versions of the lectures were also delivered at TASI 2004 and DARS 2006 cosmology schools. The selection of topics reflects author's personal idiosyncrasies as to what are the important and interesting issues at the crossroads of theoretical cosmology and theoretical high energy physics, at the beginning of the 'Golden Age' of cosmology. This is at the time after the discovery of cosmic acceleration and just before the era of LHC. How these choices are judged may change after the impending discoveries in the next decade. The author hopes that some of the topics may still remain of interest, if for no better reason than as an illustration of how we advanced the field. The students who courageously took the course from which these notes emerged deserve much credit, in particular Damien Martin (who TeXed the first installment of the lectures and generated many of the figures) and Derrick Kiley. Many errors were avoided because the author's friends and collaborators set him straight. For this, they deserve author's thanks, in particular Nima Arkani-Hamed, Christos Charmousis, Csaba Csáki, Gia Dvali, Ruth Gregory, Simeon Hellerman, Alberto Iglesias, Manoj Kaplinghat, Matt Kleban, Ian Kogan, Albion Lawrence, John March-Russell, Slava Mukhanov, Rob Myers, Tony Padilla, Minjoon Park, Steve Shenker, Eva Silverstein, Martin Sloth, Lorenzo Sorbo, John Terning and Jun'ichi Yokoyama. The author owes a debt of gratitude to his teachers over the years, Savas Dimopoulos, Andrei Linde, Keith Olive, and Lenny Susskind. Any errors which may have remained in this text are solely the author's responsibility. After all, perfection can only be found in the quest for it.

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Chapter 1

1.1 Basic Facts About Our Universe (as seen by a particle theorist)

The dominant force in the universe at large scales is gravity. The key reason behind this is that although gravity is feeble compared to the other fundamental forces currently believed to exist in Nature (see Table 1.)), it is *not* get screened, as there are no negative energy objects. By the equivalence principle, qravity couples to *total* energy which is always positive, for localised particles, and therefore cumulative: energy≡ rest mass. The absence of isolated negative energy objects is an empirical fact, which is promoted into one of the cornerstones of all meaningful frameworks of microscopic physics, being needed to ensure vacuum stability. Indeed, in quantum mechanics we must have stable ground states to build predictive deterministic frameworks, governed by unitarity. Perhaps such frameworks may exist even if we relax the requirement that spectra be bounded from below. However, until this date meaningful quantum theories with spectra that are not bounded form below have not appeared, and taking this as circumstantial evidence in what follows we will assume that no such exist.

Force	Mediator	g	M (GeV)	Range (cm)
E&M	A_μ	1/137	0	∞
Weak	$W^\pm_\mu$, Z_μ	10^{-5}	80–90	10^{-16}
Strong	g^a_μ	1	0	$< 10^{-13}$
Gravity	$h_{\mu\nu}$	10^{-38}	0	∞

Table 1: Fundamental forces in Nature: coupling, mediator mass and range.

So gravity remains while other forces fade away. This is a blessing and a curse.

Blessing: gravity is a weak force and so it is mostly clean from nonlinear and strong coupling complications.

Curse: It is an *UNSTABLE* force system. Gravitating systems tend to either fall apart or collapse – because of the weakness of the force it takes a long time and so we get a chance to understand some aspects of gravity – but the universe is an old and big place and this leads to a mystery: why has it hung around for so long? We will deal with an explanation later.

1.1.1 Description of gravity

Newton (a long time ago):

$$m\ddot{\vec{x}} = -\nabla V \tag{1.1}$$

$$V = -G_N \frac{mM}{|\vec{x}|} \tag{1.2}$$

$$G_N = 6.67 \times 10^{-11} \text{N m}^2/\text{kg}^2 \tag{1.3}$$

Central potential force

- Conservation of energy!
- Elliptical orbits in a plane (requires $F \propto r^{-2}$, not just central force)!

Proof of planar motion

The angular momentum is defined by

$$\vec{L} = \vec{r} \times \vec{p} = m\vec{x} \times \dot{\vec{x}} \tag{1.4}$$

where $\vec{r} = \vec{x}$. Differentiating gives

$$\dot{\vec{L}} = \cancel{m\dot{\vec{x}} \times \dot{\vec{x}}} + m\vec{x} \times \ddot{\vec{x}} = 0 \tag{1.5}$$

where the last term vanishes because for a central force law $\vec{x}$ is parallel to $\ddot{\vec{x}}$.

This shows $\vec{L}$ is constant $\Leftrightarrow$ orbit confined in a plane.

Finding the equations of motion

The Lagrangian for a system we have

$$L = m\left(\frac{\dot{r}^2}{2} + \frac{1}{2}r^2\dot{\varphi}^2\right) - V(r) \tag{1.6}$$

φ is a cyclic variable, so we have

$$mr^2\dot{\varphi} = m\ell = \text{constant} \Rightarrow \dot{\varphi} = \frac{\ell}{r^2} \tag{1.7}$$

To solve, introduce the *Routhian*, which is defined as a Laplace transform over cyclic variables

$$R \equiv L - \dot{\varphi}\frac{\partial L}{\partial \dot{\varphi}} \tag{1.8}$$

$$= m\left(\frac{\dot{r}^2}{2} - \frac{\ell^2}{2r^2}\right) - V(r) \tag{1.9}$$

Using the specific form of the Newtonian potential we have

$$R = m\left(\frac{\dot{r}^2}{2} - \frac{\ell^2}{2r^2}\right) + G_N\frac{Mm}{r} \tag{1.10}$$

The Routhian still obeys the r Euler-Lagrange equation, which leads to the following equation of motion

$$\ddot{r} = \frac{\ell^2}{r^3} - \frac{G_N M}{r^2} \tag{1.11}$$

Using the result that $\mathrm{d}x/\mathrm{d}\varphi = \dot{x}/\dot{\varphi}$. For the second derivatives

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\varphi^2} = \frac{1}{\dot{\varphi}}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{r}}{\dot{\varphi}}\right) \tag{1.12}$$

$$= \frac{\ddot{r}}{\dot{\varphi}^2} - \frac{\dot{r}\ddot{\varphi}}{\dot{\varphi}^3}, \quad \ddot{\varphi} = -2\frac{\ell}{r^3}\dot{r} \tag{1.13}$$

where the expression on the RHS comes from differentiating the angular momentum equation (1.7). Putting this together with (1.11) we have

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\varphi^2} = \frac{r^4}{\ell^2}\left(\frac{\ell^2}{r^3} - \frac{G_N M}{r^2}\right) + \frac{2\ell}{r^3}\frac{\cancel{\dot{\varphi}^2}}{\cancel{\dot{\varphi}^3}}\left(\frac{\mathrm{d}r}{\mathrm{d}\varphi}\right)^2 \tag{1.14}$$

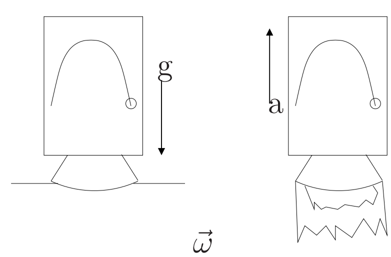
$$\therefore \frac{\mathrm{d}^2 r}{\mathrm{d}\varphi^2} - \frac{2}{r}\left(\frac{\mathrm{d}r}{\mathrm{d}\varphi}\right)^2 = r - \frac{G_N M}{\ell^2}x^2 \tag{1.15}$$

1.1.2 Problems with Newtonian gravity

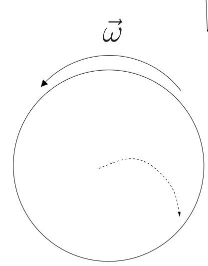
1. Perihelion of Mercury!
2. Bending of light rays
3. Radar echo delays
4. ...

Mach: inertial mass vs gravitational mass. The question of absolute versus relative motion.
Newton: absolute motion
Einstein: relative motion
Forces can be "faked" by accelerated frames!

1) Falling accelerators



2) Coriolis force



1.2 Special relativity

The laws of physics are independent of the inertial frame in which they are formulated:

$$\mathrm{d}s^2 = \eta_{\mu\nu}\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu \tag{1.16}$$

Which is invariant under transformations between inertial frames:

$$x^\mu \to x^{\mu\prime} = \Lambda^\mu{}_\nu x^\nu, \qquad \eta_{\mu\nu}\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = \eta_{\alpha\beta} \tag{1.17}$$

This implies that all laws of physics are formulated as local, casual field theories invariant (covariant) under the Lorentz (Poincaré) group!

1.3 General relativity

$$\boxed{\text{gravity} \equiv \text{accelerated frame}}$$

Thus gravity "arises" from an acceleration of an observer relative to some frame where locally gravitational force is not felt!

$$\mathrm{d}s^2 = \eta_{\mu\nu}\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu \tag{1.18}$$

$$\zeta^\mu = \zeta^\mu(x^\mu) \tag{1.19}$$

x^μ: curved coordinate system!
Locally

$$\mathrm{d}\zeta^\mu = \frac{\partial \zeta^\mu}{\partial x^\alpha}\mathrm{d}x^\alpha \tag{1.20}$$

$$\therefore \mathrm{d}s^2 = \eta_{\mu\nu}\left(\frac{\partial \zeta^\mu}{\partial x^\alpha}\mathrm{d}x^\alpha\right)\left(\frac{\partial \zeta^\nu}{\partial x^\beta}\mathrm{d}x^\beta\right) \tag{1.21}$$

$$= \left(\eta_{\mu\nu}\frac{\partial \zeta^\mu}{\partial x^\alpha}\frac{\partial \zeta^\nu}{\partial x^\beta}\right)\mathrm{d}x^\alpha\, \mathrm{d}x^\beta \tag{1.22}$$

$$= g_{\alpha\beta}\, \mathrm{d}x^\alpha\, \mathrm{d}x^\beta \tag{1.23}$$

Now we are claiming that physical laws are INVARIANT under ANY changes of co-ordinate frame!

In general, we have

$$\bar{x}^\mu = \bar{x}^\mu(x^\alpha) \tag{1.24}$$

$$\bar{g}_{\mu\nu}\,\mathrm{d}\bar{x}^\mu\,\mathrm{d}\bar{x}^\nu = \bar{g}_{\mu\nu}\frac{\partial\bar{x}^\mu}{\partial x^\alpha}\frac{\partial\bar{x}^\nu}{\partial x^\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta \tag{1.25}$$

$$\equiv g_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta \tag{1.26}$$

$$\therefore g_{\alpha\beta} = \bar{g}_{\mu\nu}\frac{\partial\bar{x}^\mu}{\partial x^\alpha}\frac{\partial\bar{x}^\nu}{\partial x^\beta} \tag{1.27}$$

$$\tag{1.28}$$

Gravitational effects are "buried" in the local deformation of the metric away from flat space! Diffeomorphism group GL(4,R).

1.3.1 Conventions

Usually in particle physics we use

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & \vec{0} \\ \vec{0} & 1_{3\times 3}\end{pmatrix} \Rightarrow \mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}\vec{x}^2. \tag{1.29}$$

In GR & cosmology the prevalent convention is the opposite:

$$g_{\mu\nu} = \sum_\lambda \lambda_\lambda (e_\lambda)_\mu (e_\lambda)_\nu \tag{1.30}$$

$$\lambda_0 = -|f_0|, \quad \lambda_k = |f_k| \tag{1.31}$$

$$\text{i.e. } \mathrm{d}s^2 = -f_0\,\mathrm{d}t^2 + f_x\,\mathrm{d}x^2 + f_y\,\mathrm{d}y^2 + f_z\,\mathrm{d}z^2 + \text{off-diagonal terms} \tag{1.32}$$

$$\text{and note} \det g < 0 \tag{1.33}$$

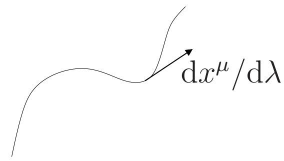
Note: We are assuming that the off-diagonal term is small. It is possible that if the OD terms are sizeable that they are all positive. We only know that three of the *eigenvalues* are positive and one is negative.

1.3.2 Particle motion

"Fermat's principle" – minimum path

$$\text{Arc length} = \int \mathrm{d}s = \int_{\lambda_a}^{\lambda_b} \sqrt{g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}}\,\mathrm{d}\lambda \tag{1.34}$$

$\frac{dx^\mu}{d\lambda}$: tangent vector to the trajectory.
λ: affine parameter.



Variation yields, for massive particles, when we can always choose $\lambda = s$ (proper time along the path)

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{ds}\frac{dx^\lambda}{ds} = 0 \tag{1.35}$$

where Γ are the *Christoffel symbols*:

$$\Gamma^\mu_{\nu\lambda} = \frac{1}{2}g^{\mu\sigma}\left(g_{\nu\sigma,\lambda} + g_{\lambda\sigma,\nu} - g_{\nu\lambda,\sigma}\right) \tag{1.36}$$

Locally there exists a coordinate system such that

$$\Gamma^\mu_{\nu\lambda}(0) = 0 \Rightarrow \frac{d^2x^\mu}{d\zeta^2} = 0 \tag{1.37}$$

Principle of equivalence: gravity $\equiv$ acceleration.

"All massive bodies fall equally"

Empirical law which is well supported by the experimental data. Precision of tests is of the order of 10^{-4}.

A note on the equivalence principle

It is probably worth noting that there are three different versions of the equivalence principles.

1. *Weak equivalence principle* (WEP)
   If an uncharged test body is placed at a specific position and let released with a specific velocity, its trajectory will be independent of the internal structure and composition.

2. *Einstein equivalence principle* (EEP)
   The WEP is valid; the outcome of any local nongravitational test is independent of the velocity of the freely falling apparatus; the outcome of any nongravitational test experiment is independent of where and when in the universe it is performed.

3. *Strong equivalence principle* (SEP)
   Hard to state precisely, but roughly states that there is no prior geometry and that gravity "gravitates" the same way as normal matter.

The interested reader is referred to the Clifford Will's book "Theory and experiment in gravitational physics".

The EEF is impossible to test directly, as it requires knowing about the everywhere at all times throughout the universe. Usually, we test the WEP directly and *assume* the other things that the EEF asserts. The best current test of the WEP is by Keiser and Faller, and the method used was floating on water. The WEP (difference in gravitational attraction for different bodies), the acceleration differs by less than one part in 10^{10}.

Chapter 2

2.1 A brief review of tensors

Tensor: a set of fields $\{T^{\mu_1 \dots \mu_n}{}_{\nu_1 \dots \nu_m}\}$ which transform linearly under a diffeomorphism.

$$x^\mu \to \bar{x}^\mu = \bar{x}^\mu(x^\nu) \tag{2.1}$$

$$\mathrm{d}x^\mu \to \mathrm{d}\bar{x}^\mu = \frac{\partial \bar{x}^\mu}{\partial x^\nu}\mathrm{d}x^\nu \tag{2.2}$$

$$\delta^\mu{}_\lambda = \frac{\partial \bar{x}^\mu}{\partial \bar{x}^\lambda} = \frac{\partial \bar{x}^\mu}{\partial x^\nu}\frac{\partial x^\nu}{\partial \bar{x}^\lambda} \tag{2.3}$$

Therefore $\frac{\partial x^\nu}{\partial \bar{x}^\lambda}$ is the inverse matrix of $\frac{\partial \bar{x}^\mu}{\partial x^\lambda}$.

Then:

$$T^{\mu_1 \dots \mu_n}{}_{\nu_1 \dots \nu_m} \to \bar{T}^{\mu_1 \dots \mu_n}{}_{\nu_1 \dots \nu_m} = \frac{\partial \bar{x}^{\mu_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial \bar{x}^{\mu_n}}{\partial x^{\alpha_n}}\frac{\partial x^{\beta_1}}{\partial \bar{x}^{\nu_1}} \cdots \frac{\partial x^{\beta_m}}{\partial \bar{x}^{\nu_m}} T^{\alpha_1 \dots \alpha_n}{}_{\beta_1 \dots \beta_m} \tag{2.4}$$

is a tensor that is n times contravariant & m-times covariant.

Damien's note: There is some tension in the GR & mathematics literature regarding covariant and contravariant. The tensors themselves are invariant under these basis changes, but the *components* are not. In GR tensors are labelled by how the components transform, while in mathematics they are classified by how basis vectors transform. To keep the actual tensor invariant these two are always opposite. We follow the GR convention here.

Examples:

ϕ	scalar	$\bar{\phi} = \phi$
V_μ	vector	$\bar{V}_\mu = \frac{\partial x^\alpha}{\partial \bar{x}^\mu} V_\alpha$
$g_{\mu\nu}$	covaraint metric:	$\bar{g}_{\mu\nu} = g_{\alpha\beta} \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \frac{\partial x^\beta}{\partial \bar{x}^\nu}$
$g^{\mu\nu}$	contravaraint metric:	$\bar{g}^{\mu\nu} = g^{\alpha\beta} \frac{\partial \bar{x}^\mu}{\partial x^\alpha} \frac{\partial \bar{x}^\nu}{\partial x^\beta}$

2.1.1 Action of derivatives

Scalars:

$$\mathrm{d}\phi = \partial_\mu \phi \, \mathrm{d}x^\mu = \bar{\partial}_\mu \phi \, \mathrm{d}\bar{x}^\mu \tag{2.5}$$

$$\Rightarrow \bar{\partial}_\mu \phi = \frac{\partial x^\nu}{\partial \bar{x}^\mu} \partial_\nu \phi \tag{2.6}$$

which transforms as a rank 1 *covariant* tensor! Therefore we define the covariant derivative of a scalar as

$$\nabla_\mu \phi \equiv \partial_\mu \phi \tag{2.7}$$

Note that the RHS of (2.7) is just notation, and the LHS is what we *mean* by it. We will use $\nabla_\mu T^{\mu_1 \ldots \mu_n}{}_{\nu_1 \ldots \nu_m}$ to denote the covariant derivative of any tensor..

Vectors:

$$\bar{V}_\mu = \frac{\partial x^\alpha}{\partial \bar{x}^\mu} V_\alpha \tag{2.8}$$

Hit with a derivative and see what happens:

$$\bar{\partial}_\nu \bar{V}_\mu = \left(\frac{\partial x^\beta}{\partial \bar{x}^\nu} \partial_\beta \right) \frac{\partial x^\alpha}{\partial \bar{x}^\mu} V_\alpha \qquad \text{(By chain rule)} \tag{2.9}$$

$$= \underbrace{\frac{\partial x^\beta}{\partial \bar{x}^\nu} \frac{\partial x^\alpha}{\partial \bar{x}^\mu} (\partial_\beta V_\alpha)}_{\text{tensor like}} + \underbrace{\frac{\partial x^\beta}{\partial \bar{x}^\nu} \left(\frac{\partial}{\partial x^\beta} \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \right) V_\alpha}_{\text{non-tensor}} \tag{2.10}$$

NOT COVARAINT! We can rewrite the offending term in a simpler way by undoing the chain rule for this term only:

$$\frac{\partial x^\beta}{\partial \bar{x}^\nu}\left(\frac{\partial}{\partial x^\beta}\frac{\partial x^\alpha}{\partial \bar{x}^\mu}\right)V_\alpha = \frac{\partial^2 x^\alpha}{\partial \bar{x}^\nu\,\partial \bar{x}^\mu}V_\alpha \tag{2.11}$$

But:

$$\bar{\Gamma}^\sigma{}_{\nu\mu} = \frac{\partial \bar{x}^\sigma}{\partial x^\alpha}\frac{\partial x^\beta}{\partial \bar{x}^\nu}\frac{\partial x^\gamma}{\partial \bar{x}^\mu}\Gamma^\alpha{}_{\beta\gamma} + \frac{\partial^2 x^\alpha}{\partial \bar{x}^\nu\,\partial \bar{x}^\mu}\frac{\partial \bar{x}^\sigma}{\partial x^\alpha} \tag{2.12}$$

So we can use this to cancel the non-tensor piece! The object

$$\nabla_\mu V_\nu = \partial_\mu V_\nu - \Gamma^\sigma_{\mu\nu}V_\sigma \tag{2.13}$$

is a derivative-like object that transforms as a rank 2 covariant tensor.

In general:

What exactly do we require of the derivative ∇_μ? By requiring the following five things, we make the definition essentially unique:

1. The derivative must be linear:
   $\nabla_\mu(\alpha A^{\mu_1\ldots\mu_n}{}_{\nu_1\ldots\nu_m} + \beta B^{\mu_1\ldots\mu_n}{}_{\nu_1\ldots\nu_m}) = \alpha\nabla_\mu A^{\mu_1\ldots\mu_n}{}_{\nu_1\ldots\nu_m} + \beta\nabla_\mu B^{\mu_1\ldots\mu_n}{}_{\nu_1\ldots\nu_m}$
   where α and β are constants.

2. The Leibnitz rule:
   $$\nabla_\mu(A^{\mu_1\ldots\mu_n}{}_{\nu_1\ldots\nu_m}B^{\rho_1\ldots\rho_p}{}_{\sigma_1\ldots\sigma_\ell}) = \left(\nabla_\mu A^{\mu_1\ldots\mu_n}{}_{\nu_1\ldots\nu_m}\right)B^{\rho_1\ldots\rho_p}{}_{\sigma_1\ldots\sigma_\ell} + A^{\mu_1\ldots\mu_n}{}_{\nu_1\ldots\nu_m}\left(\nabla_\mu B^{\rho_1\ldots\rho_p}{}_{\sigma_1\ldots\sigma_\ell}\right)$$

3. Taking the covariant derivative commutes with contraction

4. Torsion free scalar fields: $\nabla_\mu\nabla_\nu\phi = \nabla_\nu\nabla_\mu\phi$

5. That the derivative of a scalar field is the partial derivative: $\nabla_\mu\phi = \partial_\mu\phi$.

Once we require these, one can prove by induction that the covariant derivative of a general scalar field is

$$\nabla_\mu T^{\nu_1\ldots\nu_n}{}_{\sigma_1\ldots\sigma_m} = \partial_\mu T^{\nu_1\ldots\nu_n}{}_{\sigma_1\ldots\sigma_m} - \sum_{k=1}^{m}\Gamma^\rho_{\mu\sigma_k}T^{\nu_1\ldots\nu_n}{}_{\sigma_1\ldots\sigma_{k-1}\rho\sigma_{k+1}\ldots\sigma_m} + \sum_{k=1}^{n}\Gamma^{\nu_k}_{\mu\rho}T^{\nu_1\ldots\nu_{k-1}\rho\nu_{k+1}\ldots\nu_n}{}_{\sigma_1\ldots\sigma_m} \tag{2.14}$$

The induction proof is given in chapter 3 of Wald's book [2].

Covariant derivative of the metric

$$\nabla_\mu g_{\nu\lambda} = 0$$

because of metric compatibility.

2.1.2 Covariant divergence

$$\nabla_\mu V^\mu = \partial_\mu V^\mu + \Gamma^\mu_{\mu\rho} V^\rho \tag{2.15}$$

But:

$$\Gamma^\mu_{\mu\rho} = \frac{1}{2} g^{\mu\sigma} \left(g_{\mu\sigma,\rho} + \cancel{g_{\rho\sigma,\mu}} - \cancel{g_{\rho\mu,\sigma}} \right) \tag{2.16}$$

$$= \frac{1}{2} g^{\mu\sigma} g_{\mu\sigma,\rho} = \frac{1}{2}\frac{1}{g}\partial_\rho g = \frac{1}{\sqrt{g}}\partial_\rho \sqrt{g} \tag{2.17}$$

where the cancellation in (2.16) occurred because these terms were simultanously symmetric and antisymmetric in μ and σ.

Therefore, we have

$$\nabla_\mu V^\mu = \partial_\mu V^\mu + \frac{1}{\sqrt{g}}(\partial_\rho \sqrt{g}) V^\rho \tag{2.18}$$

$$= \frac{1}{\sqrt{g}} \left(\sqrt{g}\partial_\mu V^\mu + (\partial_\mu \sqrt{g}) V^\mu \right) = \frac{1}{\sqrt{g}} \partial_\mu \left(\sqrt{g} V^\mu \right)! \tag{2.19}$$

2.2 Integrability conditions

If there exist coordinates suct that $\Gamma^\mu_{\nu\lambda} = 0$ everywhere, then the spacetime is globally flat → NO GRAVITATIONAL FiELD!

(Technical note: the spacetime maybe topologically different from Minkowski, but in any local region away from a *conical singularity* is will be indistinguishable from Minkowski space.)

So gravity is hidden in the $\Gamma^\mu_{\nu\lambda}$:

$$\text{gravity} \equiv \text{deviation from flat space} \equiv \text{curvature}$$

Sufficent and necessary condition for the *existence* of a coordinate system that makes the $\Gamma^{\mu}_{\nu\lambda} = 0$ is

$$R^{\mu}{}_{\nu\lambda\sigma} \equiv 0 \tag{2.20}$$

where the *Riemann curvature tensor* is defined by

$$R^{\mu}{}_{\nu\lambda\sigma} = \partial_{\lambda}\Gamma^{\mu}_{\nu\sigma} - \partial_{\sigma}\Gamma^{\mu}_{\nu\lambda} + \Gamma^{\mu}_{\lambda\rho}\Gamma^{\rho}_{\nu\sigma} - \Gamma^{\mu}_{\sigma\rho}\Gamma^{\rho}_{\nu\lambda} \tag{2.21}$$

Gravitational field	$\Leftrightarrow$	$R^{\mu}{}_{\nu\lambda\sigma} \neq 0$
Gravity	$\Leftrightarrow$	Riemannian geometry
	$\Leftrightarrow$	Invariance under arbitrary nonlinear coordiante transformations.

Gauge theory – gauged Lorentz symmetry!
Complications: physical transformation in BOTH $g_{\mu\nu}$ and $\Gamma^{\mu}_{\nu\lambda}$.

Geodesic equation:

$$\frac{\mathrm{d}^2 x^{\mu}}{\mathrm{d}s^2} + \Gamma^{\mu}_{\nu\lambda}\frac{\mathrm{d}x^{\nu}}{\mathrm{d}s}\frac{\mathrm{d}x^{\lambda}}{\mathrm{d}s} = 0 \tag{2.22}$$

shows $\Gamma^{\mu}_{\nu\lambda}$ as the "gravitational force", and $g_{\mu\nu}$ as the "gravitational potential".

$R^{\mu}{}_{\nu\lambda\sigma}$	:	measure of gravitational field – if it is zero, spacetime is (locally) Minkowski $\Rightarrow$ no (local) gravity!

Thus gravitaional dynamics is encoded somehow in $R^{\mu}{}_{\nu\lambda\sigma}$.

Indeed: $R^{\mu}{}_{\nu\lambda\sigma}$ contains two derivatives of $g_{\mu\nu}$, and so if $g_{\mu\nu}$ is to play the role of the gravitational potential, an expression linear in $R^{\mu}{}_{\nu\lambda\sigma}$ would have the right structure to play the role of a local, causal $\rightarrow$ 2nd order in derivatives – field equation.

Hint: non-relativistic graviational field is controlled by the gravitational potential of a source M:

$$\phi = -G_N\frac{M}{|\vec{x}|} \tag{2.23}$$

which obeys the Poisson equation

$$\nabla^2\phi = -4\pi G_N M\delta^{(3)}(\vec{x}) \tag{2.24}$$

$$\therefore \qquad \nabla^2\phi = -4\pi G_N \rho(\vec{x}) \tag{2.25}$$

where $\rho(\vec{x})$ is the matter density.

Now in special relativity, ρ is component of a rank 2 stress-energy tensor $T_{\mu\nu}$, and so indeed ϕ should also transform as a 00-component of a rank 2-tensor$\Rightarrow g_{\mu\nu}$! In fact, in the weak field limit static configurations yield the Newtonian approximation

$$g_{00} = -1 + 2\phi \tag{2.26}$$

such that the Poisson equation naturally comes about if the relativistic equation is

$$G_{\mu\nu} = 8\pi G_N T_{\mu\nu} \tag{2.27}$$

where

- $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$, Einstein tensor
- $R_{\mu\nu} = R^{\mu}{}_{\nu\lambda\sigma}$, Ricci tensor
- $T_{\mu\nu}$, stress-energy tensor.

Riemann tensor obeys the Bianchi identities

$$\nabla_{(\lambda}R^{\alpha}{}_{\beta\mu\nu)} = 0 \tag{2.28}$$

where the round brackets denote symmetrisation. We can deduce

$$\nabla_{\mu}G^{\mu}{}_{\nu} = 0. \tag{2.29}$$

Then from Einstein's equations

$$\nabla_{\mu}T^{\mu}{}_{\nu} = 0. \tag{2.30}$$

This is REDUNDANT since

$$\nabla_{\mu}T^{\mu}{}_{\nu} = \sum_{\text{matter}} J_{\nu} \times (\text{field equation}) \tag{2.31}$$

Example:

Scalar field ϕ:

$$T_{\mu\nu} = \nabla_\mu\phi\,\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}(\nabla\phi)^2 - g_{\mu\nu}\frac{m^2}{2}\phi^2 \tag{2.32}$$

$$\nabla_\mu T^{\mu\nu} = \nabla^\nu\phi(\Box\phi - m^2\phi) \tag{2.33}$$

which vanishes on shell, as it is proportional to the Klien-Gordon equation.

Notice: since $g_{\mu\nu} = \mathrm{diag}(-1,\vec{1})$ the field theory convention, defined by

$$\Box\phi + m^2\phi = 0 \tag{2.34}$$

changes to, because

$$\Box = g^{\mu\nu}\partial_\mu\partial_\nu \to -g^{\mu\nu}\nabla_\mu\nabla_\nu \tag{2.35}$$

after changing metric conventions, so we have

$$\Box\phi - m^2\phi = 0 \tag{2.36}$$

instead.

The Bianchi identities are a local statement of general covariance: they ensure that there are no additional constraints on the equations of motion.

2.3 Stress-Energy tensor and the action principle

Einstein's equations can be derived from the Einstein-Hilbert action by the usual variational procedure with fixed boundaries:

$$S = \int \mathrm{d}^4x\sqrt{g}\left(\frac{R}{16\pi G_N} - \mathcal{L}_{\mathrm{matter}}\right) \tag{2.37}$$

(See physics/0504179 for an interesting account of the history of the Einstein-Hilbert action, and what the respective roles of Einstein and Hilbert were).

The stress-energy tensor $T^{\mu\nu}$ is the functional derivative of the matter action with respect to $g_{\mu\nu}$:

$$T^{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta S_{\mathrm{matter}}}{\delta g_{\mu\nu}} \tag{2.38}$$

and it is symmetric and, as we saw above conserved.

Now:

$$\delta\sqrt{g} = \frac{\sqrt{g}}{2} g^{\mu\nu}\delta g_{\mu\nu} \tag{2.39}$$

$$\delta S = \int \mathrm{d}^4 x\sqrt{g}\left(\frac{g^{\mu\nu}}{2}\frac{R}{16\pi G_N}\delta g_{\mu\nu} - \frac{1}{16\pi G_N} + \frac{1}{2}T^{\mu\nu}\delta g_{\mu\nu}\right) \tag{2.40}$$

where we have

$$\delta R = \delta(g^{\mu\nu}R_{\mu\nu}) = (\delta g^{\mu\nu})R_{\mu\nu} + g^{\mu\nu}\,\delta R_{\mu\nu} \tag{2.41}$$

and we convert the variations $\delta g^{\mu\nu}$ into ones in $\delta g_{\mu\nu}$ as follows:

$$\delta g^{\mu\nu} = -g^{\alpha\mu}g^{\beta\nu}\delta g_{\alpha\beta} \tag{2.42}$$

This allows us to write

$$(\delta g^{\mu\nu})R_{\mu}\nu = -R^{\mu\nu}\delta g_{\mu\nu} \tag{2.43}$$

Finding the variation $\delta R_{\mu\nu}$ is tedious, but some work has to be done somewhere. Using $R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}$ you should prove to yourself

$$\delta R_{\mu\nu} = \nabla_\lambda \delta\Gamma^\lambda_{\mu\nu} - \nabla_\nu\delta\Gamma^\lambda_{\mu\lambda}. \tag{2.44}$$

This implies

$$\delta R = -R^{\mu\nu}\delta g_{\mu\nu} + \nabla_\lambda(g^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu}) - \nabla_\nu(g^{\mu\nu}\delta\Gamma^\lambda_{\mu\lambda}) \tag{2.45}$$

$$= -R^{\mu\nu}\delta g_{\mu\nu} + \nabla_\mu J^\mu \tag{2.46}$$

Thus

$$\delta S = \int \mathrm{d}^4 x\sqrt{g}\left\{\left(\frac{g^{\mu\nu}}{2}\frac{R}{16\pi G_N} - \frac{R^{\mu\nu}}{16\pi G_N}\right)\delta g_{\mu\nu} + \frac{1}{2}T^{\mu\nu}\delta g_{\mu\nu} + \nabla_\mu J^\mu\right\}$$

$$= \int \mathrm{d}^4 x\sqrt{g}\left\{\left(\frac{g^{\mu\nu}}{2}\frac{R}{16\pi G_N} - \frac{R^{\mu\nu}}{16\pi G_N}\right)\delta g_{\mu\nu} + \frac{1}{2}T^{\mu\nu}\delta g_{\mu\nu} + \underbrace{\frac{1}{\sqrt{g}}\partial_\mu\sqrt{g}J^\mu}_{\text{boundary term}}\right\}$$

We assume that we can drop the boundary term (not *always* valid!), but in the cases it is we obtain the Einstein field equations as the stationary points in the action:

$$G^{\mu\nu} = 8\pi G_N T^{\mu\nu} \tag{2.47}$$

Chapter 3

Lecture 3

3.1 Cosmological solutions

Universe is extremely homogenous and isotropic at very large scales. WHY? Density deviations are of the order 10^{-5}.

To leading order we can approximate the geometry by the metric which has homogenous and isotropic spatial sections.

Homogenity & isotropy	$\rightarrow$	invariance under spatial traslations and rotations 6 generators

These geometries have maximally symmetric subspaces (see Weinberg [1]).

Depending on the spatial curvature, there are 3 allowed transitive geometries, parameterised by k:

Flat	Spherical	Hyperbolic
k =0	1	-1

The metric can be put into the form

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2 \left(\frac{\mathrm{d}r^2}{1-kr^2} + r^2 \mathrm{d}\Omega_2^2 \right) \tag{3.1}$$

Symmetries leave only one function undetermined: $a(t)$, the cosmological scale factor. This is determined by Einstein's equations. The spatial symmetries then also restrict the form of the stress-energy tensor.[1] It is characterised by 4 eigenvalues; indeed at any point p we can diagonalise $T^{\mu}{}_{\nu}$ such that

$$T^0{}_0 = -\rho; \qquad T^k{}_k = p_k; \quad \text{others} = 0 \tag{3.2}$$

i.e.

$$T^{\mu}{}_{\nu} = \left(\begin{array}{c|ccc} -\rho & & 0 & \\ \hline & p_1 & 0 & 0 \\ 0 & 0 & p_2 & 0 \\ & 0 & 0 & p_3 \end{array}\right)^{\mu}{}_{\nu} \tag{3.3}$$

The spatial symmetries then ensure that this diagonalisation remains true everywhere, since the symmetry group acts transitively, mapping every point to every other point. In particular p_1, p_2 and p_3 cannot be functions of x (homogeneity). Rotational invariance requires no special directions, so $p_1 = p_2 = p_3$:

$$T^{\mu}{}_{\nu} = \begin{pmatrix} -\rho & 0 \\ 0 & p\mathbb{1} \end{pmatrix}^{\mu}{}_{\nu} \tag{3.4}$$

We will often deal with simple fluids, obeying a simple linear (barotropic) equation of state $p = w\rho$

$p = 0$	non-relativistic gas
$p = \rho/3$	relativistic gas
$p = -\rho/3$	cosmic strings
$p = -\rho$	cosmological constant

If we define the comoving velocity vector u_μ, such that a particle at a fixed $\vec{x}$ is at rest relative to it, then in the FRW coordinates

$$u_\mu = (1, \vec{0}) \tag{3.5}$$

so that

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu} \tag{3.6}$$

Einstein tensor: by symmetries $G^0{}_0$ and $G^k{}_k$!

[1] Damien: The logic here seems backwards to me. The *observations* tell us to a good degree of accuracy that the stress-energy has the form (3.4), from which we infer the FRW metric.

Note:

$$\frac{\mathrm{d}r^2}{1-kr^2} + r^2\mathrm{d}\Omega_2^2 = \frac{\mathrm{d}r^2}{1-kr^2} - \mathrm{d}r^2 + \mathrm{d}r^2 + r^2\mathrm{d}\Omega_2^2 \tag{3.7}$$

$$= \mathrm{d}\vec{x}^2 + \frac{1-1+kr^2}{1-kr^2}\mathrm{d}r^2 \tag{3.8}$$

$$= \mathrm{d}\vec{x}^2 + \frac{k}{1-k\vec{x}^2}(\vec{x}\cdot\mathrm{d}\vec{x})^2 \tag{3.9}$$

$$= \left(\delta_{k\ell} + k\frac{x_\ell x_k}{1-k\vec{x}^2}\right)\mathrm{d}x^k\,\mathrm{d}x^\ell \tag{3.10}$$

$$\therefore \quad g_{k\ell} = \delta_{k\ell} + k\frac{x_\ell x_k}{1-k\vec{x}^2} \tag{3.11}$$

$$g^{k\ell} = \delta^{k\ell} - kx^k x^\ell \tag{3.12}$$

The Christoffel symbols are

$$\Gamma^0_{00} = \Gamma^0_{0k} = \Gamma^k_{00} = 0 \tag{3.13}$$

$$\Gamma^n_{0k} = \frac{1}{2}g^{np}g_{kp,0} = H\delta^n{}_k, \qquad H \equiv \frac{\dot{a}}{a} \tag{3.14}$$

$$\Gamma^0_{k\ell} = -\frac{1}{2}g^{00}g_{k\ell,0} = -\frac{1}{2}2a\dot{a}\left(\delta_{k\ell} + k\frac{x_\ell x_k}{1-k\vec{x}^2}\right) = -Hg_{k\ell} \tag{3.15}$$

$$\Gamma^m_{k\ell} = \frac{1}{2}g^{mn}(g_{kn,\ell} + g_{\ell n,k} - g_{k\ell,n}) = kx^m g_{k\ell} \tag{3.16}$$

3.1.1 Energy conservation:

We know that

$$\nabla_\mu T^{\mu\nu} = 0 \tag{3.17}$$

This constrains ρ and p in the following way

$$\nabla_\mu T^\mu{}_\nu = 0 = \partial_\mu T^\mu{}_\nu + \Gamma^\mu_{\mu\lambda}T^\lambda{}_\nu - \Gamma^\rho_{\mu\nu}T^\mu{}_\rho \tag{3.18}$$

Let us look at the spatial parts:

$$\nabla_\mu T^\mu{}_k = \partial_\mu T^\mu{}_k + \Gamma^\mu_{\mu\lambda}T^\lambda{}_k - \Gamma^\rho_{\mu k}T^\mu{}_\rho \tag{3.19}$$

$$= \cancel{\partial_k p} + p\Gamma^\mu_{\mu k} - p\Gamma^\rho_{\rho k} \equiv 0 \tag{3.20}$$

where the first term vanishes by homogeneity. The "conservation of energy" is non-trivial:

$$\nabla_\mu T^\mu{}_0 = 0 = \partial_\mu T^\mu{}_0 + \Gamma^\mu_{\mu\lambda} T^\lambda{}_0 - \Gamma^\rho_{\mu 0} T^\mu{}_\rho \tag{3.21}$$

$$= \dot{T}^0{}_0 + \Gamma^\mu_{\mu 0} T^0{}_0 - \Gamma^k_{k0} T^k{}_k \tag{3.22}$$

$$= -\dot{\rho} - 3H(p+\rho) \tag{3.23}$$

Therefore:

$$\dot{\rho} + 3H(\rho + p) = 0 \tag{3.24}$$

i.e. *adiabatic evolution!*

3.1.2 Ricci tensor

In general we have

$$R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu} = \partial_\lambda \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\mu\lambda} + \Gamma^\lambda_{\lambda\rho}\Gamma^\rho_{\mu\nu} - \Gamma^\lambda_{\nu\rho}\Gamma^\rho_{\mu\lambda} \tag{3.25}$$

For FRW:

$$R_{00} = R^\lambda{}_{0\lambda 0} = R^k{}_{0k0} = \cancel{\partial_k \Gamma^k_{00}} - \partial_0 \Gamma^k_{0k} + \cancel{\Gamma^k_{k\rho}\Gamma^\rho_{00}} - \Gamma^k_{0\rho}\Gamma^\rho_{0k} \tag{3.26}$$

$$= -3\dot{H} - H^2 \delta^k_\ell \delta^\ell_k = -3(\dot{H} + H^2) \tag{3.27}$$

So $R^0{}_0 = 3(\dot{H} + H^2)$.

Similarily:

$$R^k{}_\ell = \left(\dot{H} + 3H^2 + 2\frac{k}{a^2}\right)\delta^k{}_\ell \tag{3.28}$$

3.1.3 Ricci scalar

$$R = R^0{}_0 + R^k{}_k = 3(\dot{H} + H^2) + 3\left(\dot{H} + 3H^2 + 2\frac{k}{a^2}\right) \tag{3.29}$$

$$= 6\dot{H} + 12H^2 + 6\frac{k}{a^2} \tag{3.30}$$

3.1.4 Einstein tensor

$$G^{\mu}{}_{\nu} = R^{\mu}{}_{\nu} - \frac{1}{2}\delta^{\mu}{}_{\nu}R \tag{3.31}$$

$$G^{0}{}_{0} = -3H^2 - 3\frac{k}{a^2} \tag{3.32}$$

$$G^{k}{}_{k} = -2\dot{H} - 3H^2 - \frac{k}{a^2} \tag{3.33}$$

3.1.5 Einstein's equations in FRW

Define the Planck mass: $8\pi G_N = M_p^{-2}$
Indeed, note that taking $c = \hbar = 1$, since

$$V = -G_N\frac{mM}{|\vec{x}|} \rightarrow [G_N] = \frac{[V][\vec{x}]}{[m]^2} = \frac{1}{[m]} \tag{3.34}$$

Therefore

$$3H^2 + 3\frac{k}{a^2} = \frac{\rho}{M_p^2} \tag{3.35}$$

$$2\dot{H} + 3H^2 + \frac{k}{a^2} = -\frac{p}{M_p^2} \tag{3.36}$$

Total system:

$$\dot{\rho} + 3H(p+\rho) = 0 \tag{3.37}$$

$$3H^2 + 3\frac{k}{a^2} = \frac{\rho}{M_p^2} \tag{3.38}$$

$$2\dot{H} + 3H^2 + \frac{k}{a^2} = -\frac{p}{M_p^2} \tag{3.39}$$

Note: take first derivative of (3.38) and eliminate $\dot{\rho}$ using (3.37):

$$6H\dot{H} - 6\frac{k}{a^2}H = -3H\frac{(p+\rho)}{M_p^2} \tag{3.40}$$

$$\therefore 2\dot{H} - 2\frac{k}{a^2} = -\frac{p}{M_p^2} - \frac{\rho}{M_p^2} \tag{3.41}$$

$$= -\frac{p}{M_p^2} - 3H^2 - 3\frac{k}{a^2} \tag{3.42}$$

$$\therefore 2\dot{h} + 3H^2 + \frac{k}{a^2} = -\frac{p}{M_p^2} \tag{3.43}$$

which is (3.39)!

So (3.37), (3.38), (3.39) are not independent! (3.38) is a 1st integral of (3.39)! ⇒ can drop (3.39).

Why? Energy conservation! ⇒ Bainchi identites → imply the vanishing of an integration constant.

Mechanical analogy: consider a particle in true "funny" dimensions (with $m = 1$)

$$L = \frac{\dot{x}^2}{2} - \frac{\dot{y}^2}{2} - V(x) \tag{3.44}$$

V conservative → $\dot{H} = 0$, where

$$H = \frac{\dot{x}^2}{2} - \frac{\dot{y}^2}{2} + V(x) \to H = E \tag{3.45}$$

1st integral: $\frac{\dot{x}^2}{2} - \frac{\dot{y}^2}{2} + V(x) = E$.

If there exists a reparameterisation invariance $t \to T(t)$ which does not change physics, we must have $E \equiv 0$ → evolution is governed by $Ht = Et$, and reparameterisation invariance requires t independence ⇒ $E = 0$.

This is precisely what happens in gravity, since t-invariance is a part of the general coordinate invariance! Then the Hamiltonian system above is precisely the limit of Einstein theory in isotropic and homogenous cosmologies: mini-superspace approximation!

So drop equation (3.39).

Chapter 4

Lecture 4

4.1 Solutions of cosmologies

$$\dot{\rho} + 3H(\rho + p) = 0 \tag{4.1}$$

$$3H^2 + 3\frac{k}{a^2} = \frac{\rho}{M_p^2} \tag{4.2}$$

$$p = w\rho \tag{4.3}$$

4.1.1 k=0

Then

$$3H^2 = \frac{\rho}{M_p^2} > 0 \tag{4.4}$$

$\rightarrow \underbrace{H > 0}_{\text{expanding}}$ or $\underbrace{H < 0}_{\text{collaping}}$.

Metric is spatially flat

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2 \mathrm{d}\vec{x}^2 \Rightarrow \text{Topology:} \mathbb{R} \times \mathbb{R}^3 = \mathbb{R}^4 \tag{4.5}$$

Expansion value depends on the equation of state:

$$\dot{\rho} + 3(1+w)H\rho = 0 \tag{4.6}$$

$$\Rightarrow \frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a} \Rightarrow \rho = \frac{\rho_0}{a^{3(1+w)}} \tag{4.7}$$

$$3H^2 = \frac{\rho}{M_p^2} = \frac{\rho_0}{M_p^2}\frac{1}{a^{3(w+1)}} = 3\left(\frac{\dot{a}}{a}\right)^2 \tag{4.8}$$

$$\therefore a^{3w+1}\dot{a}^2 = \frac{\rho_0}{3M_p^2} = C^2 \tag{4.9}$$

So:

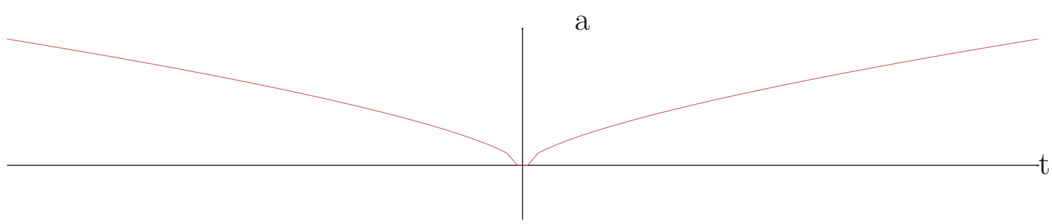
$$a^{\frac{3w+1}{2}}\dot{a} = \pm C \tag{4.10}$$

Case 1) $3w+1>0$

$$a^{\frac{3(w+1)}{2}} = \pm\tilde{C}t \tag{4.11}$$

$$\therefore \quad a(t) = (\pm\tilde{C}t)^{\frac{2}{3(w+1)}} \tag{4.12}$$

In this case $\ddot{a} < 0$, so the universe is *decelerating*.

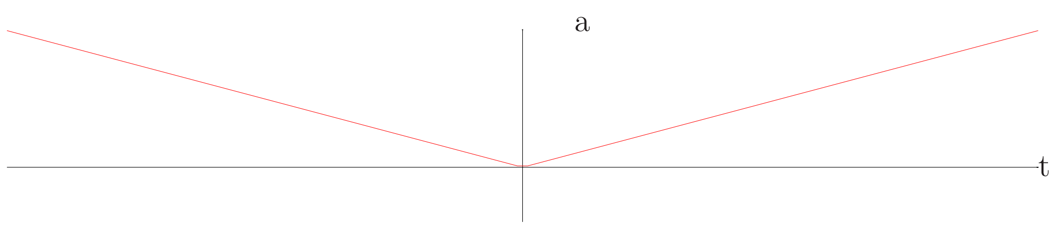


Case 2) $3w+1=0$

$$a = \pm Ct \tag{4.13}$$

$$\therefore \quad a(t) = (\pm\tilde{C}t)^{\frac{2}{3(w+1)}} \tag{4.14}$$

In this case $\ddot{a} = 0$, so the universe is *coasting*.

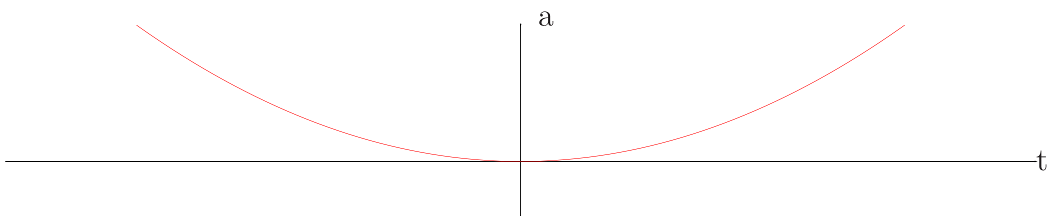


Case 3) $-2 < 3w + 1 < 0$

$$3w + 3 < 2 \Rightarrow \frac{2}{3(w+1)} > 1 \tag{4.15}$$

$$\therefore \quad a(t) = (\pm\tilde{C}t)^{\frac{2}{3(w+1)}} \tag{4.16}$$

In this case $\ddot{a} > 0$, so the universe is *accelerating*

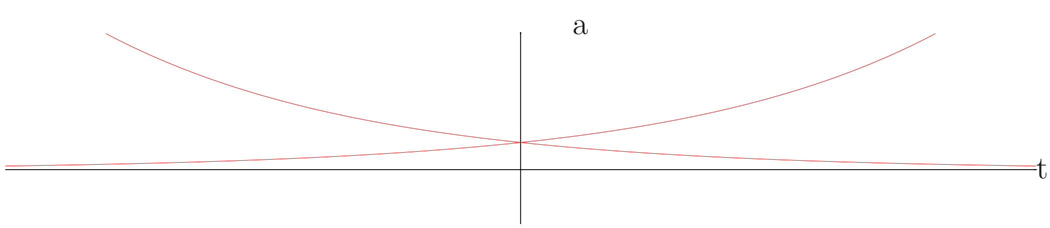


Case 4): $w = -1$

$$\frac{\dot{a}}{a} = \pm C \tag{4.17}$$

$$a(t) = a_0 e^{\pm Ct} \tag{4.18}$$

de Sitter – to be discussed more later.

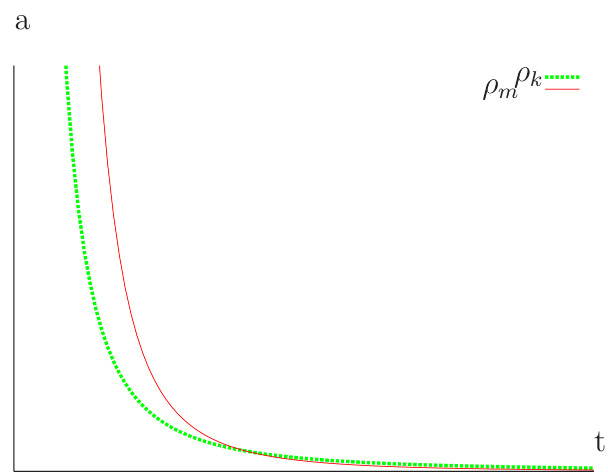


4.1.2 $k = -1$, spatially open

$$3H^2 = \frac{\rho}{M_p^2} + \frac{3}{a^2} \tag{4.19}$$

Similar to flat case: H is positive or negative so the universe either collapses or expands; the topology is $\mathbb{R} \times \mathbb{H}_3$.

Depending on the equation of state, the universe can have asymptotic evolution towards curvature dominated era: if $w > -1/3$, $\rho \sim a^{-\alpha}$, $\alpha > 2$. So as $a \gg 1$, ρ becomes subdominant to $1/a^2$:

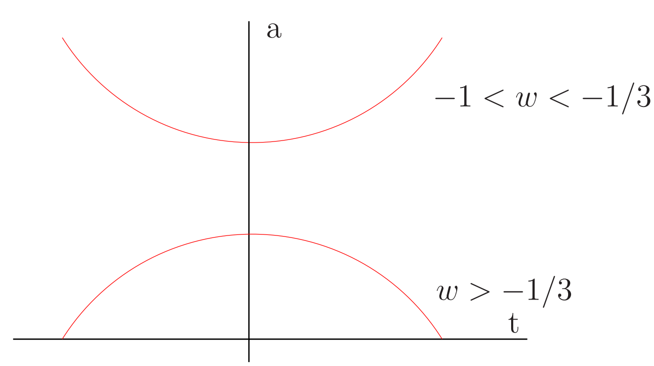


(Green is curvature and red is matter density).

4.1.3 $k = 1$: spatially closed universe

$$3H^2 = \rho - \frac{3}{a^2} \tag{4.20}$$

More interesting: now H can start with one sign, go through zero and change sign on the other side of the extremum (of a). So the universe either starts out expanding, reaches the maximum size and recollapses, or it starts out with infinite size, collapses until it reaches the minimum size and the bounces back out.



What are the asymptotic regions? Consider

$$R = 6\dot{H} + 12H^2 + 6\frac{k}{a^2} \tag{4.21}$$

Eliminate $\dot{H}$ and H^2 by equations of motion:

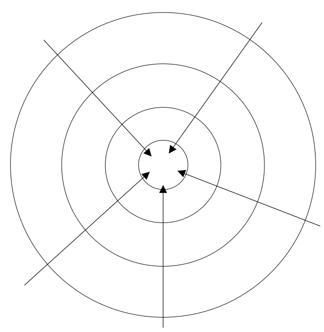
$$R = \frac{\rho - 3p}{M_p^2} \tag{4.22}$$

By $p = w\rho$ and $\rho = \rho_0 a^{-3(1+w)}$, we see

$$R = \frac{1-3w}{M_p^2}\frac{\rho_0}{a^{3(1+w)}} \tag{4.23}$$

1) $w > -1$: R blows up as $a \to 0$
2) $w > -1$: R constant

$w > -1 \to \rho + p = (1+w)\rho > 0$
Null energy condition: region of vanishing a is curvature singularity $\to$ the curvature $R \sim a^{-\alpha^2}$ diverges there; physical picture.[1]



Energy gets squeezed to tiny regions of space, back reacting violently on the geometry.

DESCRIPTION BASED ON EINSTIEN'S GRAVITY COMPLETELY BREAKS DOWN!

[1]Damien: It is not clear from this statement how the null energy condition is being invoked. NK is presumably referring to the cosmological singularity theorems. In this context I think the strong energy condition is being invoked.

We cannot extend through the singular region using the standard theory of gravity since the corrections are NOT under control:

$$S_{\text{eff}} = \int \mathrm{d}^4x\sqrt{g}\left(\frac{M_p^2}{2}R + aR^2 + b\frac{R^3}{M_p^2} + \dots\right) \tag{4.24}$$

where R^2, R^3/M_p^2, ... are LARGE!

The singularity is ALWAYS present in the $k = 0$, $k = -1$ cases because of the monotonicity of $a(t)$ ($H > 0$ or $H < 0$).

In the $k = 1$ case, the singularity may be unreachable because of the bounce: universe starts out infinite, with $a \to \infty$ and $R \to 0$, collapses until it hits $H = 0$ at some a_{min}, and expands back out to infinity.

However, this kind of singularity "removal" is unstable! A small initial perturbation with energy density $\rho \sim a^{-n}$ can get "blueshifted" (grows as a decreases) and prevent the universe from collapsing.

More work needed

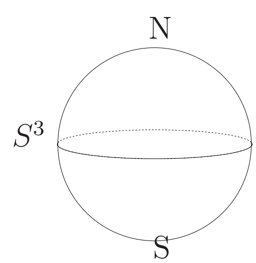
4.1.4 Special $k = 1$ case

Suppose ρ and $3/a^2$ *exactly* cancel each other in the H^2 and $\dot{H}$ equations, so that $H = 0$:

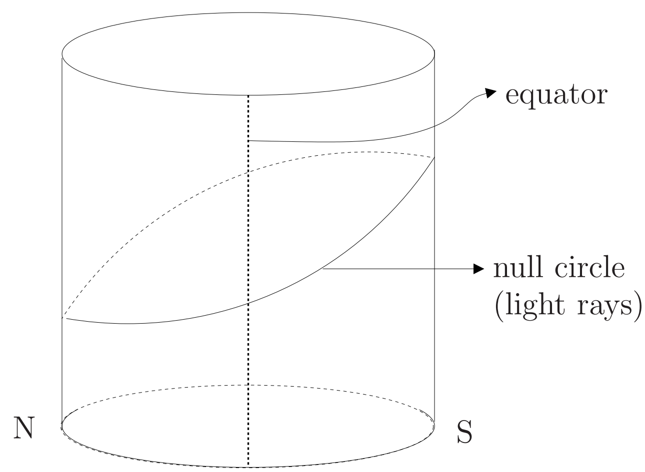
$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a_0^2\left(\frac{\mathrm{d}r^2}{1-r^2} + r^2\mathrm{d}\Omega_2^2\right) \tag{4.25}$$

$$\text{i.e.}\quad \mathrm{d}s^2 = -\mathrm{d}t^2 + a_0^2\mathrm{d}\Omega_3^2 \tag{4.26}$$

Einstein static universe: a spatially spherical geometry of constant radius.



Topology: $\mathbb{R} \times S^3$:="cylinder"



Note: light rays emitted from the North pole first diverge, until they reach the equator – after that they start to reconverge and get focussed! "Lensing" purely by topology.

Equator ≡ apparent horizon

which is the surface of vanishing geodesic expansion (vanishing geodesic "divergence").

In practise, consider the FRW geometry:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2\left(\frac{\mathrm{d}r^2}{1-kr^2} + r^2\mathrm{d}\Omega_2^2\right) \tag{4.27}$$

Change coordinates to $R = ar$:

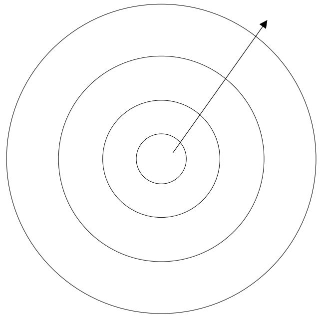
$$\mathrm{d}r = \frac{1}{a}(\mathrm{d}R - HR\mathrm{d}t) \tag{4.28}$$

$$\mathrm{d}s^2 = -\frac{1-\left(\frac{k}{a^2}+H^2\right)R^2}{1-\frac{k}{a^2}R^2}\mathrm{d}t^2 + \frac{\mathrm{d}R^2}{1-\frac{k}{a^2}R^2} - 2\frac{HR\,\mathrm{d}R\,\mathrm{d}t}{1-\frac{k}{a^2}R^2} + R^2\mathrm{d}\Omega_2^2 \tag{4.29}$$

$$\mathrm{R} = \text{const}: \quad n_\mu \mathrm{d}x^\mu = \mathrm{d}R \Rightarrow n_\mu = (0,1,0,0) \tag{4.30}$$

where n_μ is the spatial normal to the surface R=const.

$$|n|^2 = g^{\mu\nu}\partial_\mu R\,\partial_\nu R = g^{RR} = \frac{1-\left(\frac{k}{a^2}+H^2\right)R^2}{(1-\frac{k}{a^2}R^2)^2} \tag{4.31}$$



Area of the two-sphere measures geodesic deviation: as long as the area increases along the null geodesics, a neighbourhood geodesics "moves" further apart, and vice versa. If area decreases, geodesics move closer together!

Apparent horizon ≡ extremal area along a geodesic family!

$$A = 2\pi R^2 \tag{4.32}$$

λ: affine parameter: $\mathrm{d}A/\mathrm{d}\lambda = 0$. i.e.

$$\frac{\mathrm{d}R}{\mathrm{d}\lambda} = 0 \tag{4.33}$$

But: if A does not change along the direction of a geodesic, that may only happen if geodesics move

1. LOCALLY *along* it! This means that A *contains* null geodesics → that its normal must also be null!

2. LOCALLY *about* it! So A does not contain null geodesics, but δn is null –coordinates are such that the normal is also null!

$$\therefore R_{\mathrm{AH}} = \frac{1}{\sqrt{\frac{k}{a^2} + H^2}} \tag{4.34}$$

or, by Friedmann equation,

$$R_{\mathrm{AH}} = \sqrt{\frac{3}{\rho}} M_p \tag{4.35}$$

For the Einstein static universe, $H = 0$, $k = 1$:

$$R_{\mathrm{AH}} = a_0 \to r_{\mathrm{AH}} = 1 \tag{4.36}$$

On the 3-sphere, from

$$\frac{\mathrm{d}r^2}{1 - r^2} = \mathrm{d}\chi^2 \tag{4.37}$$

$$\Rightarrow r = \sin\chi, \quad r_{\mathrm{AH}} = 1 \Leftrightarrow \chi_{\mathrm{AH}} = \frac{\pi}{2} \tag{4.38}$$

Therefore apparent horizon ≡ equator!

Apparent horizon controls the evolution of gravitational instabilities!

Note: Jeans radius is defined as the linear distance defining the volume of space which for a fixed environmental density contains enough mass to be gravitationally bound:

$$R \sim R_{\mathrm{BH}} \frac{M}{M_P^2} \sim \frac{\rho}{M_P^2} R^3 \rightarrow R^2 \sim \frac{M_P^2}{\rho} \tag{4.39}$$

$$\therefore R \sim \frac{M_p}{\sqrt{\rho}} \sim R_{\mathrm{AH}} \tag{4.40}$$

See Carr & Hawking, [5].

Bottom line: the apparent horizon divides the spacetime into two regions: in one, the "interior", both past and future oriented geodesics are divergent. In the other, "the exterior", at least one of the families is convergent.

TRAPPED REGION	future-oriented geodesics are convergent
ANTITRAPPED REGION	past-oriented geodesics are convergent

Examples:

TRAPPED	*black hole horizon
	*big crunch
ANTITRAPPED	*spatial future in eternal universe.

Chapter 5

Lecture 5

5.1 Causal structure

(References for this chapter are Hawking and Ellis [4] and Helleman, NK and Susskind [6]).

The simplest way to understand the geometric properties at a world with a metric $g_{\mu\nu}$, and in particular the causal relations between events, is to analyse it with the help of light rays, i.e. null geodesics.

In the instance the geometry possesses symmetries there is a very powerful technique one can use –conformal maps.

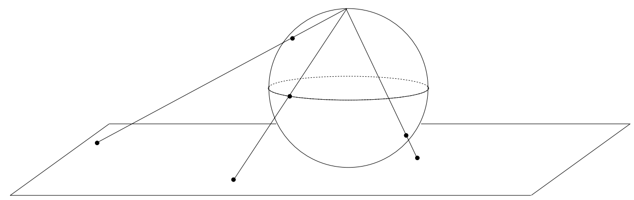
The key: map is full, infinite spacetime on a compact space, so that everything is in "front of one's eyes".

5.1.1 Carter-Penrose diagrams

Due to B. Carter, popularised by R. Penrose.

Idea: map "infinity" onto a finite boundary by a conformal rescaling.

In Euclidean geometry, this is the inverse of a stereographic projection:



Note that the null geodesics remains UNAFFECTED by conformal transformations. Indeed:

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}s^2} = \Gamma^\mu_{\nu\lambda}\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\frac{\mathrm{d}x^\nu}{\mathrm{d}s} = 0 \tag{5.1}$$

$$g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\frac{\mathrm{d}x^\nu}{\mathrm{d}s} = 0 \tag{5.2}$$

Now define

$$\bar{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} \tag{5.3}$$

The new Chistoffel symbols are

$$\bar{\Gamma}^\mu_{\nu\lambda} = \Gamma^\mu_{\nu\lambda} + \delta^\mu{}_\nu \partial_\lambda \ln\Omega + \delta^\mu{}_\lambda \partial_\nu \ln\Omega - \bar{g}_{\nu\lambda}\bar{g}^{\mu\rho}\partial_\rho \ln\Omega \tag{5.4}$$

and redefine $s \to \bar{s} = f(s)$ such that $\mathrm{d}\bar{s} = \Omega^2 \mathrm{d}s$. Then

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}\bar{s}} = \frac{1}{\Omega^2}\frac{\mathrm{d}x^\mu}{\mathrm{d}s} \tag{5.5}$$

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\bar{s}^2} = \frac{1}{\Omega^2}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\Omega^2}\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\right) \tag{5.6}$$

$$= \frac{1}{\Omega^4}\left(\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}s^2} - 2\frac{\mathrm{d}\ln\Omega}{\mathrm{d}s}\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\right) \tag{5.7}$$

So:

$$\begin{aligned}\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\bar{s}^2} + \bar{\Gamma}^\mu_{\nu\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\bar{s}}\frac{\mathrm{d}x^\lambda}{\mathrm{d}\bar{s}} = \frac{1}{\Omega^4}\Bigg(&\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}s^2} - \cancel{2\frac{\mathrm{d}\ln\Omega}{\mathrm{d}s}\frac{\mathrm{d}x^\mu}{\mathrm{d}s}} + \Gamma^\mu_{\nu\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}s}\frac{\mathrm{d}x^\lambda}{\mathrm{d}s}\\ &+ \cancel{2\frac{\mathrm{d}\ln\Omega}{\mathrm{d}s}\frac{\mathrm{d}x^\mu}{\mathrm{d}s}} - g_{\nu\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}s}\frac{\mathrm{d}x^\lambda}{\mathrm{d}s}\left(g_{\nu\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}s}\frac{\mathrm{d}x^\lambda}{\mathrm{d}s}\right)\Bigg)\end{aligned}$$

If we are looking at a null *path* in the metric $g_{\mu\nu}$ (and hence in $\bar{g}_{\mu\nu}$ as well, as they are proportional) the last term vanishes. This leaves us with

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\bar{s}^2} + \bar{\Gamma}^\mu_{\nu\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\bar{s}}\frac{\mathrm{d}x^\lambda}{\mathrm{d}\bar{s}} = \frac{1}{\Omega^4}\left(\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}s^2} + \Gamma^\mu_{\nu\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}s}\frac{\mathrm{d}x^\lambda}{\mathrm{d}s}\right) \tag{5.8}$$

i.e. If this is a null geodesic in $g_{\mu\nu}$, then the bracket on the RHS is *zero*. This implies that the geodesic equation is satisfied in $\bar{g}_{\mu\nu}$ also! So conformal transformations do not change null geodesics!

5.1.2 Minkowski space

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}\vec{x}^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\mathrm{d}\Omega_2^2 \tag{5.9}$$

in spherical polar coordinates. Define null coordinates

$$u = \frac{r-t}{\sqrt{2}}, \qquad v = \frac{r+t}{\sqrt{2}} \tag{5.10}$$

$$\Rightarrow \mathrm{d}u\,\mathrm{d}v = \frac{\mathrm{d}r^2 - \mathrm{d}t^2}{2} \tag{5.11}$$

so

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v + r^2(u,v)\mathrm{d}\Omega_2^2, \quad r \equiv \frac{u+v}{\sqrt{2}} \tag{5.12}$$

Define two more variables by

$$u = \tan\xi, \qquad v = \tan\zeta \tag{5.13}$$

To cover the original ranges of t and r we require $-\pi/2 \leq \zeta, \xi \leq \pi/2$. The differentials change by

$$\mathrm{d}u = \frac{\mathrm{d}\xi}{\cos^2\xi}, \qquad \mathrm{d}v = \frac{\mathrm{d}\zeta}{\cos^2\zeta} \tag{5.14}$$

Therefore the line element becomes

$$\mathrm{d}s^2 = 2\frac{\mathrm{d}\zeta\,\mathrm{d}\xi}{\cos^2\zeta\,\cos^2\xi} + \frac{1}{2}\left(\tan\zeta + \tan\xi\right)^2\mathrm{d}\Omega_2^2 \tag{5.15}$$

$$= \frac{1}{\cos^2\zeta\,\cos^2\xi}\left(2\mathrm{d}\zeta\,\mathrm{d}\xi + \frac{1}{2}(\sin\zeta\cos\xi + \sin\xi\cos\zeta)^2\mathrm{d}\Omega_2^2\right) \tag{5.16}$$

$$= \frac{1}{\cos^2\zeta\,\cos^2\xi}\left(2\mathrm{d}\zeta\,\mathrm{d}\xi + \frac{1}{2}\sin^2(\zeta+\xi)\,\mathrm{d}\Omega_2^2\right) \tag{5.17}$$

Now use *yet another* coordinate transformation:

$$R = \frac{\xi+\zeta}{\sqrt{2}}, \qquad T = \frac{\xi-\zeta}{\sqrt{2}} \tag{5.18}$$

$$\Rightarrow \xi = \frac{R+T}{\sqrt{2}}, \qquad \zeta = \frac{R-T}{\sqrt{2}}, \qquad \mathrm{d}\zeta\mathrm{d}\xi = \frac{\mathrm{d}R^2 - \mathrm{d}T^2}{2} \tag{5.19}$$

In terms of the new coordinates

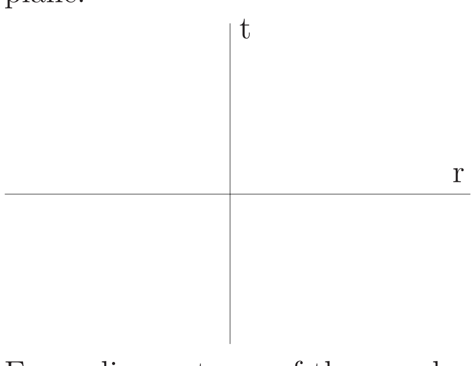
$$ds^2 = \frac{1}{\cos^2\left[\frac{R+T}{\sqrt{2}}\right]\cos^2\left[\frac{R-T}{\sqrt{2}}\right]}\left(-dT^2 + dR^2 + \frac{1}{2}\sin^2(\sqrt{2}R)d\Omega_2^2\right) \quad (5.20)$$

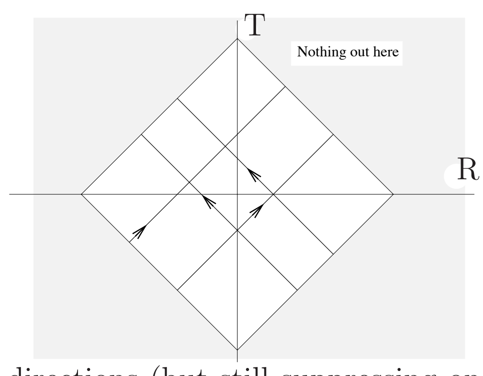
The original coordinates r, t in terms of R and T are given by

$$r = \frac{1}{\sqrt{2}}\left(\tan\frac{R-T}{\sqrt{2}} + \tan\frac{R+T}{\sqrt{2}}\right) \quad (5.21)$$

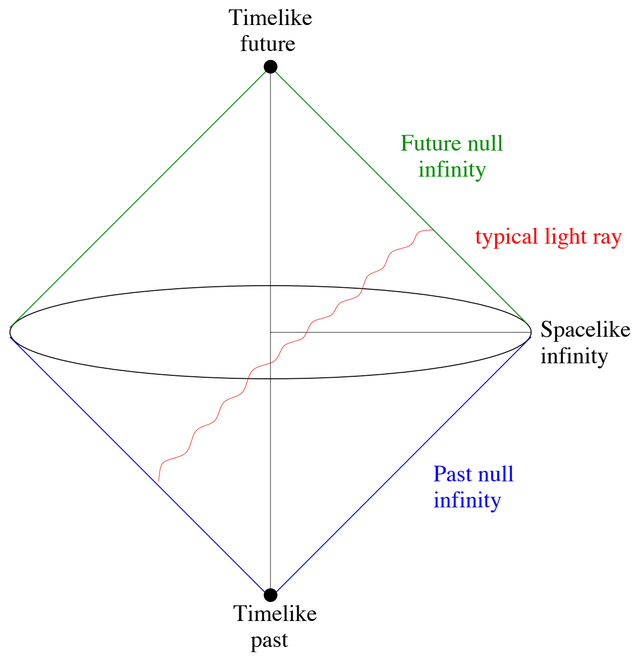
$$t = \frac{1}{\sqrt{2}}\left(\tan\frac{R-T}{\sqrt{2}} - \tan\frac{R+T}{\sqrt{2}}\right) \quad (5.22)$$

In order to cover the whole r–t plane we only need a finite region of the R–T plane.

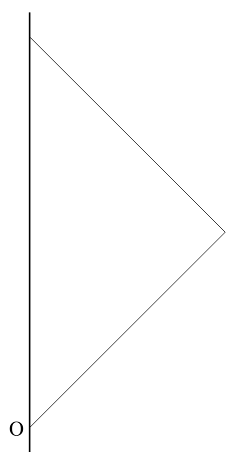




Expanding out one of the angular directions (but still suppressing one) this leads to the following picture which is *conformal* to Minkowski spacetime



5.1.3 Horizon



Suppose O launches a missile radially outward at the speed of light – the fastest affordable speed at which communication is possible in special & general relativity.

If the signal has been emitted at time t_0, by time t it can only reach observers out to distance $L = ct$ in Minkowski space. Similarly, if observations begun at time t_0, only events up to distances $L = ct$ can by seen at time t.

$\therefore$ limits to causal communication

Take a positivist point of view: only things which can be seen have a physical meaning – what can never be seen is essentially irrelevant!!!

How can we determine what can and what cannot be seen (i.e. be relevant)?

In relativistic cosmologies

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2 \left(\frac{\mathrm{d}r^2}{1 - kr^2} + r^2 \mathrm{d}\Omega_2^2 \right) \tag{5.23}$$

universe:

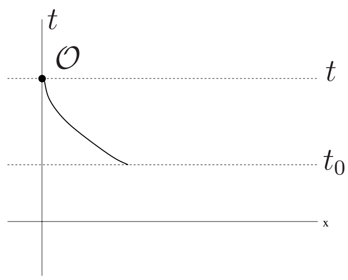
1. Expands
2. start out at a singularity

1. Because it expands, it can happen that objects far away begin to recede "faster than the speed of light". i.e. regions of the universe are so far away that because of rapid expansion they never get a chance to communicate with each other
2. Because there is an initial singularity, some objects were not initially in causal contact, and as time goes on more and more end up communicating with each other.

Particle horizon and event horizon are measures of past and future limits of causal communication.

Null geodesics:

$$\frac{\mathrm{d}t}{\mathrm{d}\lambda} = \frac{a(t)}{\sqrt{1 - kr^2}} \frac{\mathrm{d}r}{\mathrm{d}\lambda} \tag{5.24}$$



How far into the past can O see? If a signal were emitted at t_0 by the time it reaches 0 at t, $r = 0$ it has travelled the comoving distance

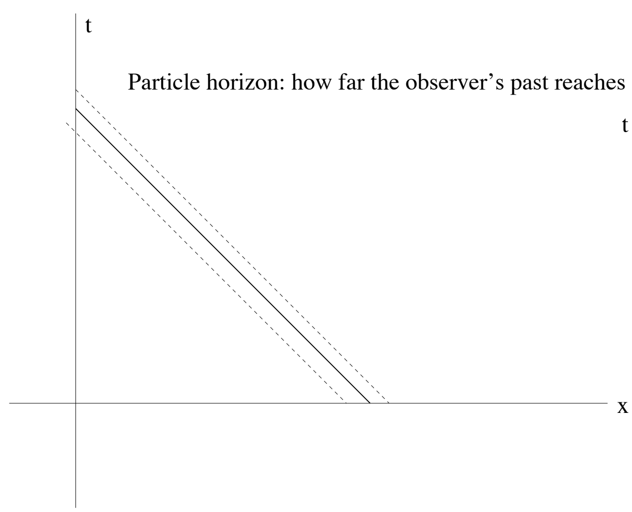
$$\int_{t_0}^{t} \frac{\mathrm{d}t'}{a(t')} = \int_0^r \frac{\mathrm{d}r}{\sqrt{1-kr^2}} = \begin{cases} r & k=0 \\ \sinh^{-1} r & k=-1 \\ \sin^{-1} r & k=+1 \end{cases} \tag{5.25}$$

This is at a proper distance

$$L_H = a(t)\chi \tag{5.26}$$

from O at time t.

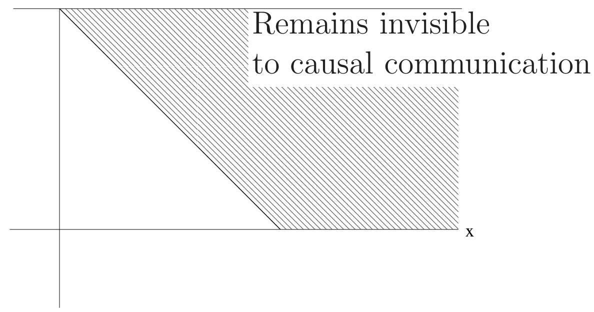
So since the universe starts out at a singularity at $t = t_0$, by time t only a finite portion of the universe will have been seen:



$$L_H = a(t) \int_0^t \frac{\mathrm{d}t'}{a(t')} \tag{5.27}$$

It grows as t grows!

Suppose now that the observer asks a different question: what is the events that can *ever* be accessed? This defines the EVENT HORIZON. The event horizon at t_0 is the intersection of the event horizon and the hypersurface $t = t_0$.



END OF TIME: $t \to \infty$

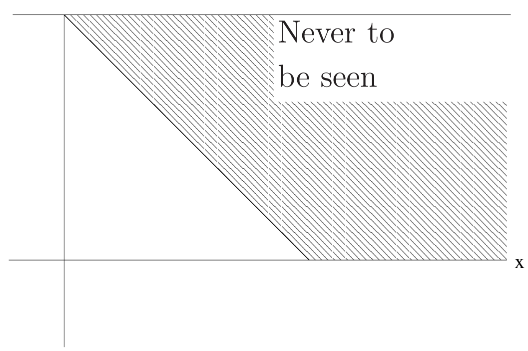
$$\int_{t_0}^{\infty} \frac{\mathrm{d}t'}{a(t')} = \chi \qquad \text{(maybe divergent)} \tag{5.28}$$

$$\therefore \qquad d_H = a(t_0)\chi \tag{5.29}$$

Future event horizon

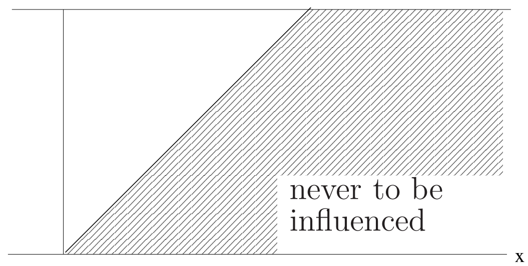
If d_H is a convergent function, then

FUTURE EVENT HORIZON $\equiv$ sup $d_H(t)$



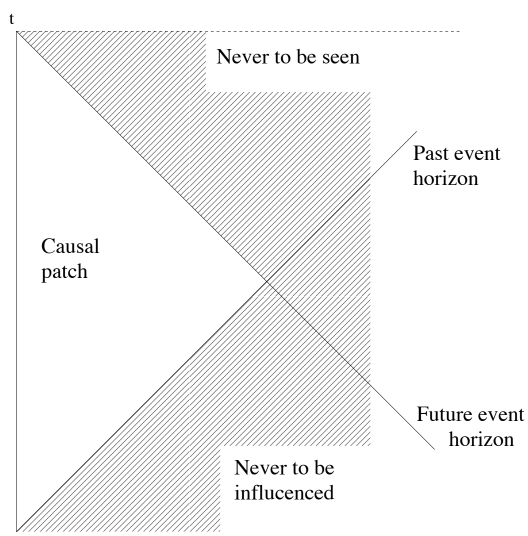
Past event horizon

Maximal 'sphere of influence"



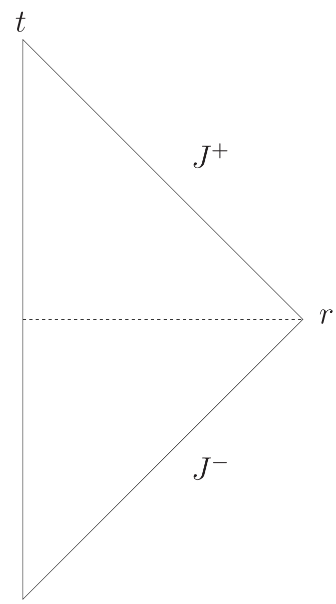
$$L_H = a(t) \int_0^t \frac{\mathrm{d}t'}{a(t')} = \text{size of particle horizon} \tag{5.30}$$

5.1.4 Causal patch



Causal patch: the set of all actions whose effects can be observed!

Example: Minkowski space



No event horizons since infinities are null (and in the right place). No geodesic starting at the centre crosses J^+or J^-in finite range of affine parameter – Minkowski patch is causal and geodesically complete!

5.1.5 Summary of horizons

This section is a summary of mine, and Namanja deserves no blame for mistakes here!

Horizon	Need to know	Use
(Future) event horizon	causally connected region(CCR) for all times	Tells us about the region that we can possibly reach.
(Past) event horizon	CCR for all times	Tells us about the region that we can ever influence.
Particle horizon at t_0	CCR up to t_0	Tells us the region we could have seen at t_0.
Apparent horizon	local region only!	Tells us about the growth of perturbations

The apparent horizon's role has not been emphasised so far, but it will be expounded upon in §11.2.

Chapter 6

Lecture 6

6.1 Causal structure of FRW universes

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2 \left(\frac{\mathrm{d}r^2}{1 - kr^2} + r^2 \mathrm{d}\Omega_2^2 \right) \tag{6.1}$$

$$3H^2 + 3\frac{k}{a^2} = \frac{\rho}{M_p^2} \tag{6.2}$$

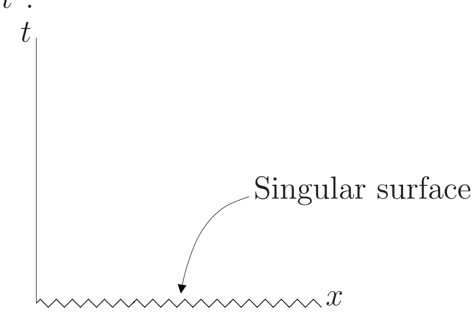
$$\rho(t) = \rho_0 \left(\frac{a_0}{a} \right)^{3(1+w)}, \qquad w = \frac{p}{\rho} \tag{6.3}$$

Example with $k = 0$

In this case

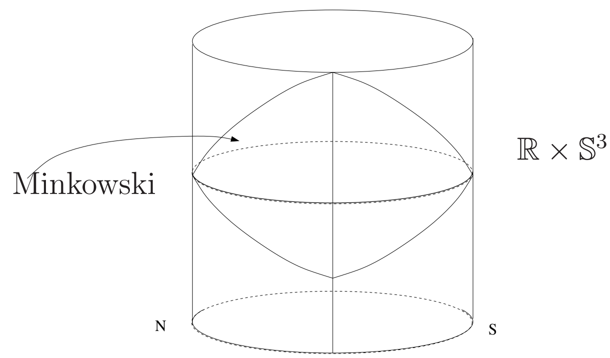
$$a = a_0 \left(\frac{t}{t_0} \right)^{\frac{2}{3(1+w)}} \tag{6.4}$$

$t = 0$ singularity: $R \sim 1/t^2$:



What is the causal structure?

Technique: Note that in the Minkowski space case, we found the causal structure by mapping the Minkowksi space into a section of the static Einstein universe:



We repeat the same trick for all FRW cosmologies!

Start with $k = 0$ cases first:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2 \mathrm{d}\vec{x}^2 \tag{6.5}$$

Notice that they are all conformally flat: change the variable to conformal time η:

$$\mathrm{d}t = a(t)\,\mathrm{d}\eta \tag{6.6}$$

$$\therefore \quad \eta = \int \frac{\mathrm{d}t}{a(t)} \tag{6.7}$$

such that

$$\mathrm{d}s^2 = a^2(-\mathrm{d}\eta^2 + \mathrm{d}\vec{x}^2) \tag{6.8}$$

1. So the spatially flat FRW cases are conformal to Minkowski.

2. We know how to map Minkowski on a patch of the Einstein static universe $\mathbb{R} \times \mathbb{S}^3$.

3. Combine the two maps!

$$\mathrm{Map}(\mathrm{FRW} \to \mathbb{R} \times \mathbb{S}^3) \cong \mathrm{Map}(M_4 \to \mathbb{R} \times \mathbb{S}^3) \circ \mathrm{Map}(\mathrm{FRW} \to M_4)$$

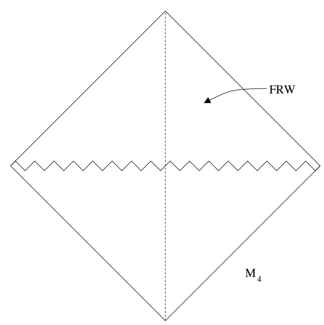
Guess:

$$a = a_0 \left(\frac{t}{t_0}\right)^{\frac{2}{3(1+w)}} \tag{6.9}$$

$$R = 6\dot{H} + 12H^2 \sim \frac{1}{t^2} \tag{6.10}$$

all FRW universes are singular at $t = 0$.

So since the Minkowski time runs over $(-\infty, \infty)$ and FRW only over $(0, \infty)$, FRW should map on half M_4; one "expects" something like



This is the correct equation of state for $w > -1/3$ – see Hawking and Ellis [4]

$w \leq -1/3$ is somewhat surprising, because in this case there are event horizons! Indeed, consider

$$d_H = a(t) \int_t^\infty \frac{\mathrm{d}t}{a(t)} \tag{6.11}$$

with

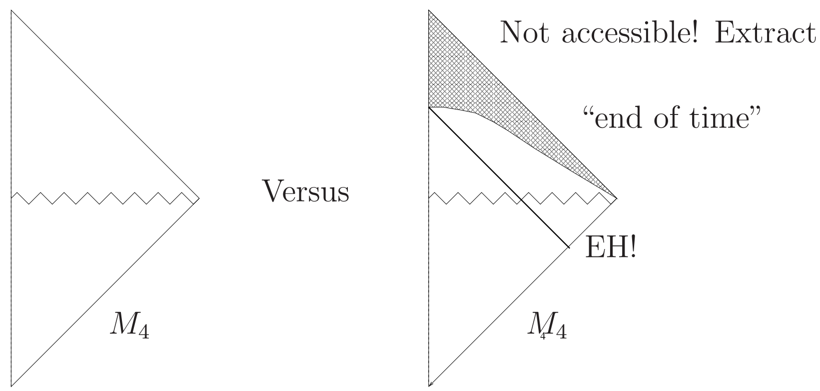
$$a(t) = a_0 \left(\frac{t}{t_0}\right)^{\frac{2}{3(1+w)}} \tag{6.12}$$

$$\Rightarrow d_H = t^{\frac{2}{3(1+w)}} \int_t^\infty \frac{\mathrm{d}t}{t^{\frac{2}{3(1+w)}}} \tag{6.13}$$

$$= \begin{cases} \frac{3(1+w)}{1+3w} t^{\frac{2}{3(1+w)}} x^{\frac{1+3w}{3(1+w)}} \Big|_t^\infty & w > -\frac{1}{3} \\ t \ln x|_t^\infty & w = -\frac{1}{3} \\ \frac{3(1+w)}{|1+3w|} t^{\frac{2}{3(1+w)}} x^{-\frac{|1+3w|}{3(1+w)}} \Big|_t^\infty & -1 < w < -\frac{1}{3} \end{cases} \tag{6.14}$$

- d_H diverges when $w \geq -1/3$.
- d_H CONVERGES when $w < -1/3$!

$w < -1/3$ universes have an EVENT HORIZON!



Detailed map

Let

$$a = a_0 \left(\frac{t}{t_0}\right)^{\frac{2}{3(1+w)}} \tag{6.15}$$

$$\eta = \int \frac{\mathrm{d}t}{t^{\frac{2}{3(1+w)}}} = \begin{cases} \frac{3(1+w)}{3w+1} t^{\frac{3w+1}{3(1+w)}} & w \neq -\frac{1}{3} \\ \ln t & w = -\frac{1}{3} \end{cases} \tag{6.16}$$

So:

$$\mathrm{d}s^2 = a^2(-\mathrm{d}\eta^2 + \mathrm{d}\vec{x}^2) = a^2(-\mathrm{d}\eta^2 + \mathrm{d}r^2 + r^2\mathrm{d}\Omega_2^2) \tag{6.17}$$

The next step: map the Minkowski space onto $\mathbb{R} \times \mathbb{S}^3$:

$$\eta = \frac{1}{2}\left[\tan\left(\frac{\chi+\tau}{2}\right) - \tan\left(\frac{\chi-\tau}{2}\right)\right] \tag{6.18}$$

$$r = \frac{1}{2}\left[\tan\left(\frac{\chi+\tau}{2}\right) + \tan\left(\frac{\chi-\tau}{2}\right)\right] \tag{6.19}$$

Then, when $w \neq -1/3$

$$\mathrm{d}s^2 = \ell^2 \left(\frac{6(1+w)}{|1+3w|}\right)^{\frac{4}{|1+3w|}} \frac{\left[\cos\left(\frac{\chi-\tau}{2}\right)\cos\left(\frac{\chi+\tau}{2}\right)\right]^{\frac{4}{|1+3w|}-2}}{4\sin^{\frac{4}{|1+3w|}}(|\tau|)}(-\mathrm{d}\tau^2+\mathrm{d}\chi^2+\sin^2\chi\,\mathrm{d}\Omega_2^2)$$

where ℓ is a length scale given by

$$\ell = \left[a_0 M_4^{\frac{2}{3(1+w)}}\right]^{\frac{3(1+w)}{|1+3w|}} \tag{6.20}$$

The following relationship will be useful in determining the limits of χ and τ:

$$\frac{1+3w}{6(1+w)}\left(\frac{t}{\ell}\right)^{\frac{1+3w}{3(1+w)}} = \tan\left(\frac{\chi+\tau}{2}\right) - \tan\left(\frac{\chi-\tau}{2}\right) \tag{6.21}$$

$w > -1/3$

Note that when $w > -1/3$, $1 + 3w > 1$ and therefore

$$t \sim \tan\left(\frac{\chi+\tau}{2}\right) - \tan\left(\frac{\chi-\tau}{2}\right) \tag{6.22}$$

Thus the singularity at $t = 0$ maps onto the line

$$\tan\left(\frac{\chi+\tau}{2}\right) = \tan\left(\frac{\chi-\tau}{2}\right) \tag{6.23}$$

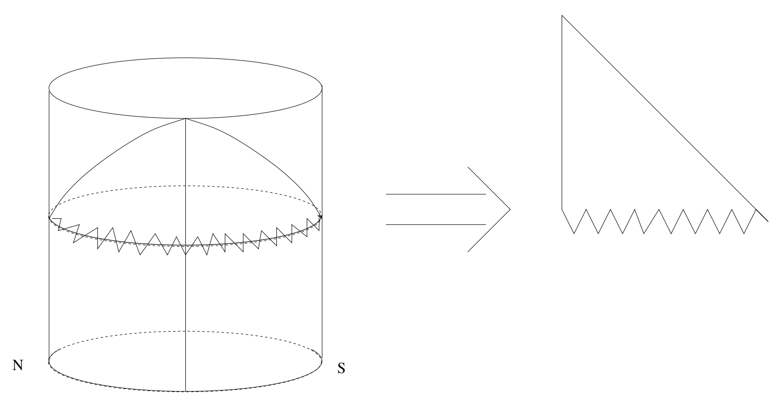
or therefore a great semicircle $\tau = 0$!

$\Rightarrow$ Singularity is spacelike

Null future infinity $\eta + r \to \infty$ maps onto

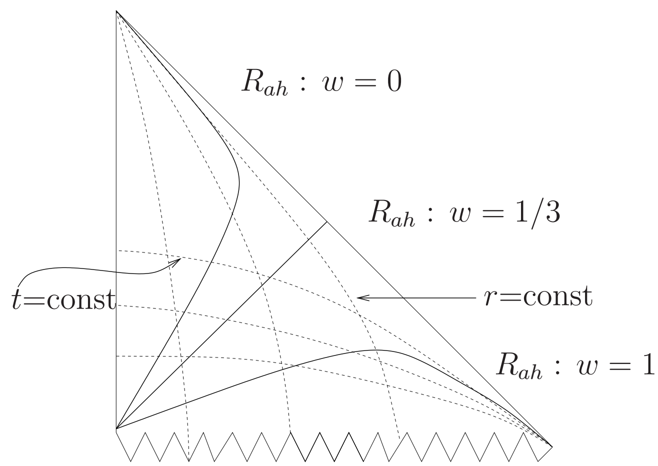
$$\tan\frac{\chi+\tau}{2} \to \infty \Rightarrow \chi + \tau = \pi \tag{6.24}$$

which is the null semicircle $\tau = \pi - \chi$. This completes the boundary map:



No event horizon; $H = \frac{2}{3(1+w)}\frac{1}{t}$. So

$$R_{ah}\ \frac{3(1+w)}{2}t \tag{6.25}$$



$w < -1/3$

When $w < -1/3$, $3w + 1 < 1$ and so

$$t \sim \frac{1}{\tan\left(\frac{\chi+\tau}{2}\right) - \tan\left(\frac{\chi-\tau}{2}\right)} \tag{6.26}$$

so this time $t \to \infty$ maps onto the great semicircle $\tau = 0$!

The singularity $t = 0$ maps onto

$$\tan\left(\frac{\chi+\tau}{2}\right) \to \infty \tag{6.27}$$

$$\text{or}\ \tan\left(\frac{\chi-\tau}{2}\right) \to \infty \tag{6.28}$$

i.e. the great 1/4 circles $\tau = \pm\chi - \pi$.

Chapter 7

Lecture 7

7.1 de Sitter space

Solution of Einstein's equations with cosmological constant

$$G_{\mu\nu} = 8\pi G_N T_{\mu\nu} = -8\pi G_N \Lambda g_{\mu\nu} \tag{7.1}$$

found by de Sitter in 1917. Looks like many things at the same time – deceiving!

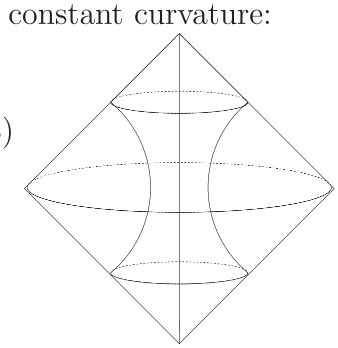
7.1.1 Construction

Consider 5D Minkowski space

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}\vec{x}_3^2 + \frac{\mathrm{d}z^2}{k} \tag{7.2}$$

and embed into it a spacetime hyperbolid of constant curvature:

$$z^2 + k(\vec{x}_3^2 - t^2) = 1 \tag{7.3}$$



Eliminate z by using the constraint

$$z\mathrm{d}z = k(t\mathrm{d}t - \vec{x}\mathrm{d}\vec{x})$$

So

$$dz^2 = \frac{k^2}{z^2}(t\mathrm{d}t - \vec{x}\mathrm{d}\vec{x})^2 \tag{7.4}$$

$$= \frac{k^2}{1-k(\vec{x}^2-t^2)}(t\mathrm{d}t - \vec{x}\mathrm{d}\vec{x})^2 \tag{7.5}$$

So note that $t\mathrm{d}t - \vec{x}\mathrm{d}\vec{x} = -\eta_{\mu\nu}\mathrm{d}x^\mu\,\mathrm{d}x^\nu$, so that

$$\mathrm{d}s^2 = \eta_{\mu\nu}\mathrm{d}x^\mu\,\mathrm{d}x^\nu + \frac{k^2}{1-k(\vec{x}^2-t^2)}(t\mathrm{d}t - \vec{x}\mathrm{d}\vec{x})^2 \tag{7.6}$$

$$= \eta_{\mu\nu}\mathrm{d}x^\mu\,\mathrm{d}x^\nu + \frac{k}{1-k\eta_{\alpha\beta}x^\alpha x^\beta}\eta_{\mu\sigma}\eta_{\mu\lambda}x^\sigma x^\lambda\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu \tag{7.7}$$

$$= \left(\eta_{\mu\nu} + \frac{k}{1-k\eta_{\alpha\beta}x^\alpha x^\beta}\eta_{\mu\sigma}\eta_{\mu\lambda}x^\sigma x^\lambda\right)\mathrm{d}x^\mu\,\mathrm{d}x^\nu \tag{7.8}$$

Thus

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{k}{1-k\eta_{\alpha\beta}x^\alpha x^\beta}\eta_{\mu\sigma}\eta_{\mu\lambda}x^\sigma x^\lambda \tag{7.9}$$

The metric has 10 isometries; let us find them. First note that from

$$\bar{g}_{\mu\nu} = g_{\alpha\beta}\frac{\partial x^\alpha}{\partial \bar{x}^\mu}\frac{\partial x^\beta}{\partial \bar{x}^\nu} \tag{7.10}$$

In the case of infinitesimal transformations

$$\bar{x}^\mu = x^\mu + \xi^\mu \tag{7.11}$$

$$\rightarrow x^\mu = \bar{x}^\mu - \xi^\mu \tag{7.12}$$

$$\frac{\partial x^\alpha}{\partial \bar{x}^\mu} = \delta^\alpha{}_\mu - \partial_\mu\xi^\alpha = \delta^\alpha{}_\mu - \xi^\alpha{}_{,\mu} \tag{7.13}$$

Note that because this is an infinitesimal transformation that we are expanding to first order, the difference between differentiating ξ^μ with ∂_μ or $\bar{\partial}_\mu$ is of second order.

So

$$\bar{g}_{\mu\nu} = g_{\alpha\beta}(\delta^\alpha{}_\mu - \xi^\alpha{}_{,\mu})(\delta^\beta{}_\nu - \xi^\beta{}_{,\nu}) \tag{7.14}$$

$$= g_{\mu\nu} - g_{\alpha\nu}\xi^\alpha{}_{,\mu} - g_{\alpha\mu}\xi^\alpha{}_{,\nu} + \mathcal{O}\left(\xi^2\right) \tag{7.15}$$

$$\therefore \bar{g}_{\mu\nu} - g_{\mu\nu} = -\left(g_{\alpha\nu}\xi^\alpha{}_{,\mu} + g_{\alpha\mu}\xi^\alpha{}_{,\nu}\right) + \mathcal{O}\left(\xi^2\right) \tag{7.16}$$

A note to all the relativity fanatics: we are only doing a coordinate transformation. The metric (as a tensorial object) *does not change*. Because we are changing coordinates, the *same* coordinates will label a *different* point and the functional form will change. So what we are really doing, in a vauge sense is comparing the metric at two nearby points from one another. (The technical term is "Lie dragging" the metric from one point to the other).

Isometry: ξ is an isometry if $g_{\mu\nu} = \bar{g}_{\mu\nu}$. i.e. if

$$g_{\alpha\nu}\xi^{\alpha}{}_{,\mu} + g_{\alpha\mu}\xi^{\alpha}{}_{,\nu} = 0 \tag{7.17}$$

or

$$\nabla_{\mu}\xi_{\nu} + \nabla_{\nu}\xi_{m}u \equiv 2\nabla_{(\mu}\xi_{\nu)} = 0 \tag{7.18}$$

i.e. ξ^{μ} is a Killing vector.

Maximally symmetric space in D dimensions $\iff \exists \frac{D(D+1)}{2}$ linearily independent Killing vectors.

Minkowski: $D(D-1)/2$ "rotations" + D translations.

7.1.2 de Sitter (dS)

In dS case, from

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}\vec{x}^2 + \frac{\mathrm{d}z^2}{k} \tag{7.19}$$

$$z^2 + k(\vec{x}^2 - t^2) = 1 \tag{7.20}$$

It is clear that the hyperboloid embedding is INVARIANT under any 5D Lorentz transformation!

$\frac{5\times4}{2} = 10$	$\frac{4\times3}{2} + 4 = 10$
5D Lorentz	4D "rotations" and "translations"

4D dS isometries are projections of the 5D Lorentz group onto the hyperboloid!

$$x^{\mu} \rightarrow x^{\mu\prime} = R^{\mu}{}_{\nu}x^{\nu} + R^{\mu}{}_{z}z \tag{7.21}$$

$$z \rightarrow z' = R^{z}{}_{\nu}x^{\nu} + R^{z}{}_{z}z \tag{7.22}$$

$$\tag{7.23}$$

1) "Rotations"

Here $R^{\mu}{}_{z} = 0$ and $R^{z}{}_{z} = 1$.

$$\eta_{\mu\nu} R^{\mu}{}_{\alpha} R^{\nu}{}_{\beta} = \eta_{\alpha\beta}, \qquad \text{Lorentz in 4D} \tag{7.24}$$
$$\Rightarrow 6 \text{ generators} \tag{7.25}$$

2) "Translations"

Here:

$$R^{\mu}{}_{z} = a^{\mu} \tag{7.26}$$
$$R^{z}{}_{\mu} = -k\eta_{\mu\nu} a^{\nu} \tag{7.27}$$
$$R^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} - bk\eta_{\alpha\nu} a^{\alpha} a^{\nu} \tag{7.28}$$
$$R^{z}{}_{z} = (1 - k\eta_{\mu\nu} a^{\mu} a^{\nu})^{1/2} \tag{7.29}$$
$$\text{with } k\eta_{\mu\nu} a^{\mu} a^{\nu} \leq 1 \tag{7.30}$$
$$\text{and } b = \frac{1 - \sqrt{1 - k\eta_{\mu\nu} a^{\mu} a^{\nu}}}{k\eta_{\mu\nu} a^{\mu} a^{\nu}} \tag{7.31}$$

Therefore

$$x^{\mu\prime} = x^{\mu} + a^{\mu} \left(\sqrt{1 - k\eta_{\bar{\mu}\bar{\nu}} x^{\bar{\mu}} x^{\bar{\nu}}} - bk\eta_{\bar{\mu}\bar{\nu}} x^{\bar{\mu}} a^{\bar{\nu}} \right) \tag{7.32}$$

To see that this solution is de Sitter, use the coordinate transformation

$$t = \frac{1}{\sqrt{k}} \left[\frac{k|\vec{x}'|^2}{2} \cosh(\sqrt{k}t') + \left(1 + \frac{k|\vec{x}'|^2}{2} \right) \sinh(\sqrt{k}t') \right] \tag{7.33}$$
$$\vec{x} = \vec{x}' e^{\sqrt{k}t'} \tag{7.34}$$
$$\mathrm{d}s^2 = -\mathrm{d}t^2 + e^{2\sqrt{k}t'} \mathrm{d}\vec{x'}^2 \tag{7.35}$$

Recall the Friedmann equation for $w = -1$, $k = 0$

$$3H^2 = \frac{\rho}{M_p^2} \tag{7.36}$$

$$\rho = \rho_0 \left(\frac{a_0}{a}\right)^{3(1+w)} = \Lambda \tag{7.37}$$

$$\therefore 3H^2 = \frac{\Lambda}{M_p^2} \to H = H_0 = \pm\sqrt{\frac{\Lambda}{3M_p^2}} \tag{7.38}$$

$$a = a_0 e^{H_0 t} \tag{7.39}$$

$$\therefore \mathrm{d}s^2 = -\mathrm{d}t^2 + a_0^2 e^{2H_0 t}\mathrm{d}\vec{x}^2 \tag{7.40}$$

Indeed, this is de Sitter with $k = H_0^2$!

Event horizon:

$$d_H = a(t)\int_t^\infty \frac{\mathrm{d}t'}{a(t')} = e^{H_0 t}\int_t^\infty \frac{\mathrm{d}t'}{e^{H_0 t}} = \frac{1}{H_0}! \tag{7.41}$$

This is also the apparent horizon. The event and apparent horizons coincide!
$\Rightarrow$ apparent horizon is a null surface!
Causal structure? Use other coordinate systems.

7.1.3 Static patch

Change coordinates to

$$\bar{t} = t' - \frac{1}{2\sqrt{k}}\ln\left(1 - k|\vec{x}|62e^{2\sqrt{k}t'}\right) \tag{7.42}$$

$$\bar{x} = x'e^{\sqrt{k}t'} = \vec{x} \tag{7.43}$$

Then the line element takes the form

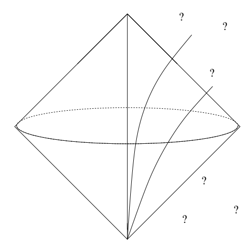
$$\mathrm{d}s^2 = -(1 - k\bar{x}^2)\mathrm{d}\bar{t}^2 + \mathrm{d}\bar{x}^2 = \frac{k(\bar{x}\cdot\mathrm{d}\bar{x})^2}{1 - k\bar{x}^2} \tag{7.44}$$

Recall $k = H_0^2$; use spherical polar coordinates

$$\mathrm{d}s^2 = -(1 - H_0^2\bar{R}^2)\mathrm{d}\bar{t}^2 + \frac{\mathrm{d}\bar{R}^2}{1 - H_0^2\bar{R}^2} + \bar{R}^2\mathrm{d}\bar{\Omega}_2^2 \tag{7.45}$$

Horizon : $\bar{R}_0 = \frac{1}{H_0}$.

Static patch coordinates cover only the region around the pole enclosed by the horizon → causal patch!



staticity is to be interpreted with care.

In a sense de Sitter geometry is really not static but STATIONARY – partilces keep flying around but the setup is in equilibrium! [1]

7.1.4 Global coordinates

Consider now $k = 1$, $w = -1$ case:

$$3H^2 + \frac{3}{a^2} = \frac{\Lambda}{M_p^2} = 3H_0^2, \qquad H = \frac{\dot{a}}{a} \tag{7.46}$$

$$\therefore \dot{a}^2 + 1 = H_0^2 a^2 \tag{7.47}$$

$$\therefore a = \frac{1}{H_0}\cosh(H_0 t) \tag{7.48}$$

$$\Rightarrow \mathrm{d}s^2 = -\mathrm{d}t^2 + \frac{1}{H_0^2}\cosh(H_0 t)\mathrm{d}\Omega_3^2 \tag{7.49}$$

or, in terms of conformal time

$$\eta = \int \frac{\mathrm{d}t}{a} = H_0 \int \frac{\mathrm{d}t}{\cosh(H_0 t)} = \int \frac{\mathrm{d}\chi}{\cosh\chi} \tag{7.50}$$

$$= 2\tan^{-1}(e^{H_0 t}) \tag{7.51}$$

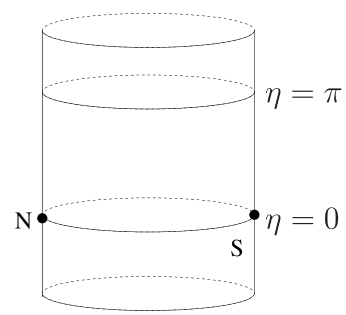
Hence $e^{H_0 t} = \tan(\eta/2)$.

[1] It should be noted that the classical case *is* static; this comment is for the real world case when you have to couple quantum fields to the metric (cf. Hawking effect)

Hence:

$$ds^2 = \frac{1}{4}\left(\tan\left(\frac{\eta}{2}\right) + \cot\left(\frac{\eta}{2}\right)\right)^2 \left(-d\eta^2 + d\Omega_3^2\right) \tag{7.52}$$

Conformal factor blows up at $\eta = 0, \pi$



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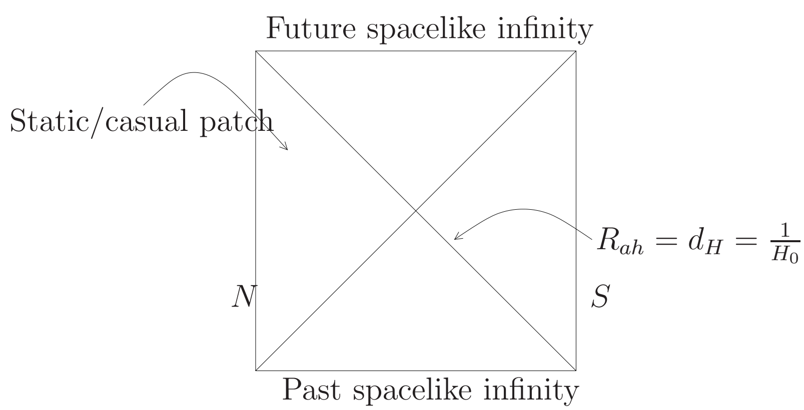
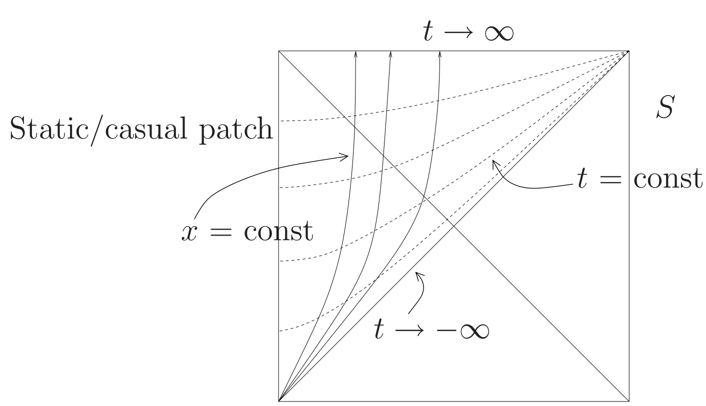




$t \to -\infty$ limit takes one to the PAST event horizon! This is where we place the casual boundary conditions $\to$ the past boundary conditions for the Cauchy problem!

Inflation is determined by boundary conditions on the past event horizon – any spacelike surface crossing it and carrying other initial data can be evolved back onto the past event horizon.

We require consistency of these initial data with conventional quantum field theory – this means, the observables carrying the information about the initial data must be the usually defined local operators of quantum field theory there. So we expect that at very short distances the theory on the past horizon should behave as a normal flat space QFT, since that is the only framework we both know and trust.

Other possibilities (i..e theories with nonlocal phenomena NOT ordinary QFT approximation at low energies) might also occur but currently they are NOT under control (or trusted!) – hence we will ignore them in what follows.

Chapter 8

Lecture 8

8.1 Cosmological problems

The universe is

- OLD: $t \sim \frac{1}{H_0} \sim 10^{10}$ years $\sim 10^{60} t_p$
- Very nearly homogeneous and isotropic: $\frac{\delta\rho}{\rho} \sim 10^{-5}$
- Nearly spatially flat $\frac{k}{a^2 H_0^2} \lesssim 10^{-2}$
- Full of the "right" kind of structure (galaxies, microwave background etc) but devoid of the "wrong" kinds (beasts: monopoles, strings, domain walls, exotic particles)
- Or it is ... Dark energy $\sim 70\%$, Dark matter $\sim 30\%$.

Cosmological problems: difficulties in reproducing the observations from arbitrary initial conditions. Two categories, by origin:

1. Gravitational
2. Particle physics

some of the problems arise from interplay of gravitational physics with particle physics.

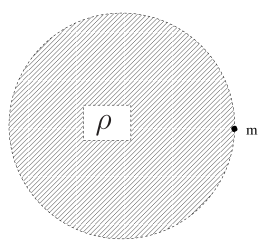
"Sources" of problems

1. Locality
2. Causality
3. Rate of expansion
4. Gravitational instability
5. Production of undesirable objects (particle physics)
6. Production of desirable objects (structure formation)

Examples of problems

1) Age and flatness

Consider a system of particles which are MARGINALLY GRAVITATIONALLY BOUND:



This means that the radius is of order of the Schwarzschild radius,

$$R_0 \sim G_N M \sim \frac{\rho}{M_p^2} R_0^3 \tag{8.1}$$

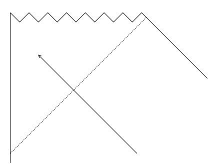
so

$$R_0 \sim \frac{M_p}{\sqrt{\rho}} \tag{8.2}$$

The geometry felt by a point particle is Schwarzschild, by Birkhoff's theorem.

$$ds^2 = -\left(1 - \frac{2G_N M}{r}\right) dt^2 + \left(1 - \frac{2G_N M}{r}\right)^{-1} dr^2 + r^2 d\Omega_2^2 \qquad (8.3)$$

Once a particle passes through the horizon it will fall all the way to the "crunch" singularity at $r = 0$; the typical infall time is



$$\tau_{\text{in}} = \int_0^{r_H} \frac{dr}{\sqrt{\frac{2G_N M}{r} - 1}} \sim G_N M \qquad (8.4)$$

$$\text{i.e. } \tau_{\text{in}} \sim R_0 \sim \frac{M_p}{\sqrt{\rho}} \sim \frac{1}{H_0} \qquad (8.5)$$

See Carr & Hawking [5].

Therefore the system is doomed to collapse in a time scale set by the initial density!

Since in Einstein's equations

$$\rho = \rho_0 \left(\frac{a_0}{a}\right)^{3(w+1)} \qquad (8.6)$$

at early times $\rho \gg \rho_{\text{now}} \to$ so the collapse time was much shorter in the past. What prevented the universe from collapsing before it reached the age of 10^{10} years?

Another way to see it:

$$3H^2 + 3\frac{k}{a^2} = \frac{\rho}{M_p^2} \qquad (8.7)$$

Define ρ_c as the energy density of a spatially flat universe for a fixed H:

$$\rho_c = 3M_p^2 H^2 \qquad (8.8)$$

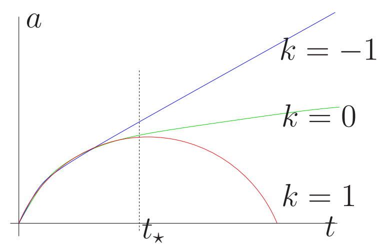
Then; a measure of spatial flatness is

$$r = \frac{3k/a^2}{\rho_c} = \frac{k}{a^2 H^2} \qquad (8.9)$$

For a radiation dominated universe, $H^2 \sim \rho_{\text{rad}} \sim a^{-4}$, while for a matter dominated universe, $H^2 \sim \rho_{\text{mat}} \sim a^{-3}$. So

$$r \sim \begin{cases} a^2 & \text{radiation} \\ a & \text{matter} \end{cases} \tag{8.10}$$

At early times r is smaller than at late times, when the universe is comprised of "normal" matter. Then, as time goes on



The turning point is roughly given by the equality $\rho_{\text{matter}} \sim a^{-2}$. Theen if $h = 1$ the age of the universe is $\sim 2t_\star$.

In the case of our own universe, observations show that

$$\frac{k}{a^2H^2} \lesssim \frac{1}{100} \tag{8.11}$$

This means that $t_\star > t_{\text{now}} \sim 10^10$years.

But this requires a *tremendous* fine tuning of the initial conditions! <u>Age problem</u>.

Note:

$$r = \frac{k}{a^2H^2} \tag{8.12}$$

Assume $\rho = \rho_0(a_0/a)^4$ for radiation. So then compare r_{now} to some initial r_0

$$\frac{r_{\text{now}}}{r_0} = \frac{a_0^2H_0^2}{a^2H^2} = \frac{a_0^2\rho_0}{a^2\rho_0\left(\frac{a_0}{a}\right)^4} \tag{8.13}$$

$$= \left(\frac{a}{a_0}\right)^2 \tag{8.14}$$

But:

$$\left(\frac{a}{a_0}\right)^2 = \sqrt{\frac{\rho_0}{\rho_{\text{now}}}} \tag{8.15}$$

Hence

$$\frac{r_{\text{now}}}{r_0} = \sqrt{\frac{\rho_0}{\rho_{\text{now}}}} \tag{8.16}$$

Now $\rho \sim (10^{-3}\text{eV})^4$, then $\rho_0 \sim (M_p)^4 \sim (10^{28}\text{eV})^4$, hence

$$\frac{r_{\text{now}}}{r_0} \sim 10^{62} \tag{8.17}$$

and so

$$r_0 \leq 10^{-64} \tag{8.18}$$

Flatness problem!

2) Horizon

Assume now that $k = 0$ to conform with observations; then for simplicity assume that the universe is radiation dominated:

$$a = a_0 \sqrt{\frac{t}{t_0}} \tag{8.19}$$

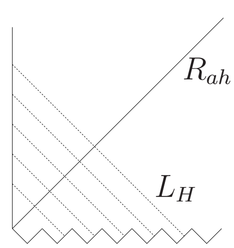
The particle horizon is

$$L_h = a(t) \int_0^t \frac{\mathrm{d}t'}{a(t')} = 2t \tag{8.20}$$

and the apparent horizon is

$$R_{ah} = \frac{1}{H} = 2t = L_H \tag{8.21}$$

So:



A typical scale of homogeneity today is given by the apparent horizon size

$$\ell \sim \frac{1}{H} \tag{8.22}$$

It scales as

$$\ell \sim \ell_0 \frac{a}{a_0} = \frac{1}{H_0}\sqrt{\frac{t}{t_0}} \tag{8.23}$$

Thus

$$\frac{\ell}{L_h} = \frac{1}{2tH_0}\sqrt{\frac{t}{t_0}} \tag{8.24}$$

$$= \frac{1}{2H_0\sqrt{tt_0}} = \sqrt{\frac{t_0}{t}} \tag{8.25}$$

At the last scattering surface, we find temperature $T_\star \sim \text{eV} \sim 1000T_0$. Since then photons moved freely, and so they carry information about the homogeneity scale then. But since

$$T = T_0\left(\frac{a_0}{a}\right) = T_0\sqrt{\frac{t_0}{t}} \tag{8.26}$$

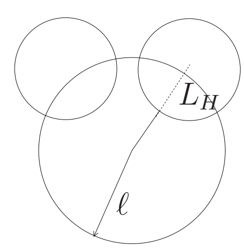
we find that

$$\frac{\ell}{L_h} = \frac{T}{T_0} \tag{8.27}$$

so at the last scattering surface

$$\frac{\ell}{L_H} \sim 10^3 \tag{8.28}$$

and so there were about $(\ell/L_h)^3 \sim 10^9$ casually disconnected domains then!



All the way down to the Planck scale

$$\frac{\ell}{L_h} \sim 10^{31} \tag{8.29}$$

→ there were about 10^{90} disconnected domains!

3) Homogeneity

See [16].

Structure likes to form over time scales set by the age of the universe. For arbitrary initial conditions that would suggest a lot more inhomogeneity and anisotropy than we see. What arrested the Jeans instability.

4) Origin of structure

Instead of the structure we see is "properly" distributed.
SCALE INVARIANT SPECTRUM.

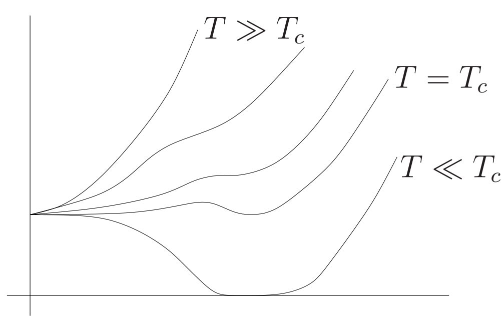
$$\frac{\delta\rho}{\rho} \sim (\ln k)^{n-1} \tag{8.30}$$

$$n - 1 \ll 1 \tag{8.31}$$

What set the right initial conditions for the formation of such structures?

5) Beasts

At very high temperatures, many particle physics symmetries are restored. As the universe cools, symmetries are spontaneously broken



Multiple vacua structure leads to the emergence of topological defects.

Monopoles, strings and domain walls – their number density should be set by the production rates governed by the Kibble mechanism. Basically, just by causality there should be one such object per Hubble volume at the time of transition. So later

$$n \sim H_\star^3 \left(\frac{a_\star}{a}\right)^3 \tag{8.32}$$

Very large energy density, since their mass scale is large!

$$\rho_{\text{monopoles}} \sim M_{\text{GUT}} H_\star^3 \left(\frac{a_\star}{a}\right)^3 \tag{8.33}$$

Why don't we see them?

Bottomline: generic initial conditions DO NOT lead to something like our universe in a natural way IF the universe is dominated by normal matter, relativistic or not!

Chapter 9

Lecture 9

9.1 Inflation: a super-cure or snake oil?

Recall again the cosmological problems:

1. Age and flatness
2. Horizon
3. Homogeneity and isotropy
4. Origin of structure
5. Beasts (monopoles, strings, domain walls, exotic particles

Also: cosmological constant, singularities, ultraviolet sensitivity…

Consider also the "sources" of problems

1. Locality
2. Causality
3. Rate of the expansion
4. Gravitational instability
5. Production of undesirable objects
6. Need to produce desirable objects

Locality and causality, and also decoupling, are fundamental conerstones of physics, which are ensuring predictivity of our algorithmic schemes to do computations.

> "Behaviour of an electron in your lab does not depend on the electrons on the moon"[1]

To date, we have NOT seen compelling evidence for action-at-a-distacne. If anything, we have accumulated a lot of evidence to the contrary. We have developed theories of relativity which are tested and found to agree with nature to great precision.

So we want to keep locality and causality; at least, until we find a predictive theoretical framework within which they could be relaxed – but in such a way that they are NOT INCONSISTENT with any observations.

Currently, we can only devise bounds on departures from locality and causality, and these suggest that any violation of locality and causality within the current framework should not play a significant role in addressing cosmological problems!

Intermezzo

How can it work? Consider the idea of HOLOGRAPHY: the information about the system is encoded on its causal boundary.

BUT MORE IMPORTANTLY: the information and energy density in the universe is LIMITED: the densest objects in the universe, both informatically and energetically, are BLACK HOLES!!!

So consider an ultra-dense universe which is TIGHTLY PACKED with black holes

1. Flatness problem:
   Equivalent to tendency of a gravitating system to collapse BUT once

[1] More strictly, we want a cluster decomposition principle. The eletrons can be effected by particles on the moon (after all, the tides are!) but it should be a small effect.

system reaches the densest possible state, it contains ALL the information it can possibly have $\rightarrow$ interactions $\iff$ exchange of information!
$\Rightarrow$ Requires NEW information!
Since there is no return for it the interactions should SHUT DOWN!
$\Rightarrow$ no more gravitational force, no more flatness problem! Similar thoughts [14], papers by Banks& Fischler.

2. Horizon problem:
   The system in the densest state may be as big as one likes: then it consists of many causally disconnected parts! Thus, while big, since each part is in its own densest state – they are all the same!
   Problem: how does one make a transition from such a holographic, dense, initial state, into a state corresponding to our very DILUTE universe where normal gravity operates?
   An alternative way to attack the flatness problem: invent arguments to enforce symmetries in the initial state, which then forces the universe to be FLAT: e.g. SUSY

Pre-Big-Bang, ekpyrotic universe: very cold initial state. Note that the initial state is ALWAYS somebody's version of Hell

Hot big bang:	Judeo-Christian Hell
Cold pre-big-bang:	Nordic Hell – NIFFELHEIM

Question: what does ALIEN HELL look like?

So we can address the cosmological problems by devising normal effective field theories, which are local and causal, where at early times the dynamics of the universe is CHANGED such that the cosmological problems are ameliorated!

1) Flatness

Recall that

$$3H^2 + 3\frac{k}{a^2} = \frac{\rho}{M_p 62} \tag{9.1}$$

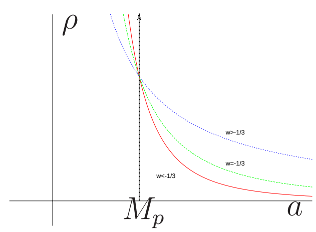
$$\rho = \rho_0 \left(\frac{a_0}{a}\right)^{3(1+w)} \tag{9.2}$$

Problem: for normal matter ($p = 0$. for non-relativistic particles, and $p = \rho/3$ for radiation) ρ dilutes FASTER than the curvature terms!

So if we assume virial initial conditions, i.e. that energy is equally distributed between all parts of the gravitating system, the curvature TAKES OVER QUICKLY! Namely, it will dominate after the universe has expanded by a factor of e, say, and a typical timescale for that is

$$\tau \sim \frac{1}{H} \qquad \text{“Lifetime”} \tag{9.3}$$

But what if we ALTER somehow the interactions so that the energy density ρ dilutes more slowly? We need $3(1 + w) < 2 \rightarrow w < -1/3$.



In that case, ρ will always dominate! Of course, such a ρ would also dominate over normal matter, so how do we eventually get the normal universe, with normal matter contents (protons, electrons, galaxies, ...) to emerge?

This strange ρ had better be UNSTABLE: it needs to DECAY into normal matter after some (presumably LONG) time!

2) Horizon

Recall that the horizon problem came from the fact that a typical homogeneity scale

$$\ell \sim \frac{1}{H_0} \frac{a}{a_0} \tag{9.4}$$

grows more slowly in a normal matter ($w = 0, 1/3$) dominated universe than the particle horizon:

$$L_H = a(t) \int_0^t \frac{\mathrm{d}t'}{a(t')} \propto t \tag{9.5}$$

But consider what happens when the equation of state is given by some general w:

$$a = a_0 \left(\frac{t}{t_0}\right)^{\frac{2}{3(1+w)}} \tag{9.6}$$

So:

$$\ell = \frac{1}{H_0}\left(\frac{t}{t_0}\right)^{\frac{2}{3(1+w)}} \tag{9.7}$$

$$L_h = t^{\frac{2}{3(1+w)}} \int_0^t \frac{\mathrm{d}t'}{(t')^{\frac{2}{3(1+w)}}} \tag{9.8}$$

$$= \begin{cases} t^{\frac{2}{3(1+w)}} \left[1 - \frac{2}{3(1+w)}\right] x^{1-\frac{2}{3(1+w)}}\Big|_{t_0}^{t} & w \neq -\frac{1}{3} \\ t \ln\left(\frac{t}{t_0}\right) & w = -\frac{1}{3} \end{cases} \tag{9.9}$$

This diverges as $t_0 \to 0$; so regulate by taking $t_0 = t_p = 1/M_p$. Thus

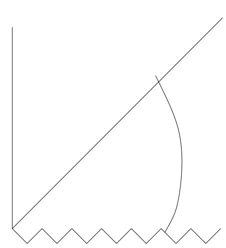
$$L_h = \begin{cases} \frac{1+3w}{3(1+w)} t & w > -\frac{1}{3} \\ t \ln(tM_p) & w = -\frac{1}{3} \\ \frac{|1+3w|}{3(1+w)} \left[\frac{(M_p t)^{\frac{2}{3(w+1)}}}{M_p} - t\right] & w < -\frac{1}{3} \end{cases} \tag{9.10}$$

So in the case $w > -1/3$:

$$\frac{\ell}{L_H} \sim \frac{t_0}{t}\left(\frac{t}{t_0}\right)^{\frac{2}{3(1+w)}} \tag{9.11}$$

$$\text{i.e. } \frac{\ell}{L_H} \sim \left(\frac{t_0}{t}\right)^{1-\frac{2}{3(1+w)}} = \left(\frac{t_0}{t}\right)^{\frac{3w+1}{3(1+w)}} \tag{9.12}$$

and ℓ increases more slowly than L_H



In the case $w < -1/3$:

$$\frac{\ell}{L_H} \sim \frac{\left(\frac{t}{t_0}\right)^{\frac{2}{3(1+w)}}}{\left(\frac{t}{t_0}\right)^{\frac{2}{3(1+w)}}} \approx \text{const} \tag{9.13}$$

Scale of homogeneity grows as fast as the horizon → homogeneity is preserved at large scales!

3) Homogeneity problem

Note from

$$2\dot{H} + 3H^2 + \frac{k}{a^2} = -\frac{p}{M_p^2} \tag{9.14}$$

$$3H^2 + 3\frac{k}{a^2} = \frac{\rho}{M_p^2} \tag{9.15}$$

$$\text{and } \frac{\ddot{a}}{a} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{a}}{a}\right) + \frac{\dot{a}^2}{a^2} = \dot{H}^2 + H^2 \tag{9.16}$$

we get

$$\frac{\ddot{a}}{a} = -\frac{1}{6M_p^2}(\rho + 3p) = -\frac{1+3w}{6M_p^2}\rho \tag{9.17}$$

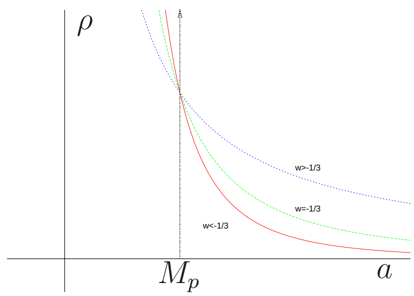
when $w < -1/3$ and $\rho > 0 \Rightarrow \ddot{a} > 0$.

ACCELERATED EXPANSION!

The Hubble parameter is given by

$$H = \frac{\dot{a}}{a} = \frac{2}{3(1+w)}\frac{1}{t} \tag{9.18}$$

when $w = -1 : H = \text{const} \equiv H_0$ (funny limit).



for $w < -1/3$, as $t \gg 1/M_p$, H approaches a constant!

So: $R_{AH} = 1/H \approx$ const. We know

$$H \gg H_0 \to R_{AH} \ll 10^{-15}m \tag{9.19}$$

Recall that R_{AH} controls formation of structure. Pertubations grow on scales $\ell \lesssim R_{AH}$ but their growth is arrested at scales $\ell > R_{AH}$. $\to$ they get blow away by accelerated expansion of the universe faster than they can form! These very short distance scales are IRRELEVANT TODAY – they are shorter than a millimetre now and have been processed many times over by subsequent non-linear evolution.

4) Structure

Structure comes from the inhomogeneities spontaneously generated by quantum perturbations.

MORE LATER

5) Beasts

Beasts are prevented from overpopulating the universe today by two effects

- Their initial density is dramatically diluted by accelerated expansion

  $$\rho \sim \frac{1}{a^3} \text{for monopoles!}$$

  $\to$ redshifts faster than the medium during accelerated expansion, $\rho \sim a^{-3(1+w)}$.

- The medium during expansion with $w < -1/3$ is designed such that when it decays, it does NOT recreate a large population of beasts!

Chapter 10

Lecture 10

10.1 Mechanics of inflation

Consider again the Friedmann equations & energy conservation

$$3H^2 + 3\frac{k}{a^2} = \frac{\rho}{M_P^2} \tag{10.1}$$

$$\dot{\rho}_k + 3H(\rho_k + p_k) = 0 \tag{10.2}$$

$$p_k = w_k \rho_k \tag{10.3}$$

$$\therefore \rho_k = \rho_{0,k}\left(\frac{a_0}{a}\right)^{3(1+w)} \tag{10.4}$$

Design an agent for whom the equation of state is such that $w \leq -1/3$. This contribution then starts to dominate quickly after the onset of the regime of validity of Einstein's equations.

10.1.1 Benchmarks

$$\frac{\ddot{a}}{a} = -\frac{1+3w}{6M_p^2}\rho > 0, \qquad \text{when } w \leq -\frac{1}{3} \tag{10.5}$$

What does inflation need to accomplish?

1) Flatness problem:

Suppose that at the onset of inflation, $\rho \sim 1/a^2$. We know that without inflation we would need a finely tuned value of

$$\frac{k}{a_*^2 H_*^2} \leq 10^{-60}$$

During inflation $H \approx H_0 \approx$ constant. So:

$$\frac{\dot{a}}{a} \approx H_0 \approx \text{constant}$$
$$\therefore a \approx a_0 \exp(H_0 \, \Delta t)$$

Thus we have

$$aH \approx a_0 H_0 \exp(H_0 \, \Delta t) \tag{10.6}$$

at the onset of inflation we have NATURAL initial condition, $aH \sim 1$. so at the end of inflation we need $k/a_*^2 H_*^2 \leq 10^{-60}$ or therefore

$$aH \geq 10^{30}$$

So

$$e^{H_0 \, \Delta t} \geq 10^{30}$$

i.e. $H_0 \, \Delta t \geq \ln 10^{30} \sim 69$.

We define the # of e-folds of inflation as an equivalent measure of the duration of inflation:

$$a = a_0 e^N \tag{10.7}$$
$$\therefore \quad N = \ln \frac{a(\text{final})}{a(\text{initial})} \tag{10.8}$$

To solve the flatness problem we need $N \geq 69$ (give or take).

2) Horizon problem

After inflation terminates we can assume that the universe becomes radiation-dominated: $T \gg m$. Then

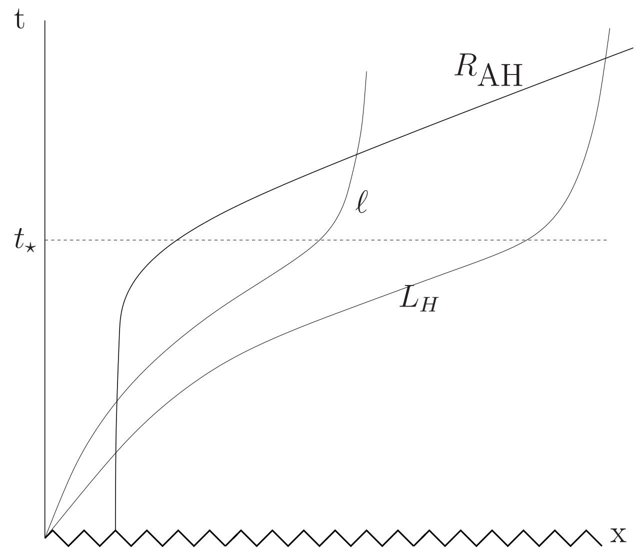
$$a = a_* \sqrt{\frac{t}{t_*}} = a_0 \sqrt{\frac{t}{t_0}} \tag{10.9}$$

So today $\ell_0 \approx 1/H_0$ and thus

$$\ell = \frac{1}{H_0}\sqrt{\frac{t}{t_0}} \sim \sqrt{t_0 t} \tag{10.10}$$

at earlier times.

$l_\star \approx \sqrt{t_0 t_*}$ is the homogeneity scale today projected into the past:



This scale MUST be smaller than or equal to the particle horizon at the end of inflation!

$$L_H = a(t_*)\int_0^{t_*} \frac{\mathrm{d}t'}{a(t')} \approx \frac{1}{H_*}e^N \approx t_* e^N \tag{10.11}$$

so at $\ell \approx \sqrt{t_0 t_*}$, $\ell \le L_H$ and

$$e^N \geq \sqrt{\frac{t_0}{t_*}} \sim 10^{30} \tag{10.12}$$

$$N \geq 69 \tag{10.13}$$

Thus $\geq$ 65–69 e-folds of inflation solve both the horizon and flatness problems! [17, 18, 19].

Blow the universe up really fast, get rid of the initial unfavourable population, and repopulate it by good stuff at the end!

10.2 Inflation

What is the agent driving inflation? Past inflationary universe should be homogenous and isotropic!

⇒No preferred directions!
⇒No preferred rest frame (no aether!?)

The object which should control the onset and termination of inflation should therefore be an invariant under rotations and translations, and while it is impossible to make it constant it should be very *weakly* time dependent.

SCALAR FIELD ≡ INFLATION![1]

Dynamics:

$$S = \int \mathrm{d}^4x\sqrt{g}\left(\frac{M_p^2}{2}R - \frac{1}{2}(\partial\phi)^2 - V(\phi)\right) \tag{10.14}$$

$$G_{\mu\nu} = \frac{1}{M_p^2}T_{\mu\nu} \tag{10.15}$$

$$\Box\phi = \frac{\partial V}{\partial\phi} \tag{10.16}$$

$$T^\mu{}_\nu = \partial^\mu\phi\partial_\nu\phi - \frac{1}{2}\delta^\mu{}_\nu(\partial\phi)^2 - \delta^\mu{}_\nu V \tag{10.17}$$

Recall that in FRW cosmologies

$$T^\mu{}_\nu = \begin{pmatrix} -\rho & \\ & p\mathbb{1} \end{pmatrix} \tag{10.18}$$

Assume that $V(\phi)$ has been designed so that it leads to inflation, and check for consistency later.

This means, that we can assume the metric is spatially flat FRW soon after the onset of inflation; this should be an excellent approximation.

Thus

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2\mathrm{d}\vec{x}^2 \tag{10.19}$$

Furthermore, homogeneity implies translational invariance, so

$$\phi = \phi(t) \to \partial_k\phi = 0 \tag{10.20}$$

Thus

$$\rho = \frac{\dot{\phi}^2}{2} + V \tag{10.21}$$

$$p = \frac{\dot{\phi}^2}{2} - V \tag{10.22}$$

$$3H^2 = \frac{1}{M_p^2}\left(\frac{\dot{\phi}^2}{2} + V\right) \tag{10.23}$$

$$\ddot{\phi} + 3H\dot{\phi} + \partial_\phi V = 0 \tag{10.24}$$

$$w = \frac{p}{\rho} = \frac{\frac{\dot{\phi}^2}{2} - V}{\frac{\dot{\phi}^2}{2} + V} \tag{10.25}$$

Need:

1. Negative pressure with $w \leq -1/3$; in fact the more negative the better
2. Inflation should be sufficiently long:

$$N = H\,\Delta t \geq 69$$

So the regime of negative pressure should be SUSTAINABLE! This implies that if $\dot{\phi}$, V are such that $w < -1/3$ it should last for a long time!

So from

$$w = \frac{\frac{\dot{\phi}^2}{2} - V}{\frac{\dot{\phi}^2}{2} + V} \tag{10.26}$$

and requiring that $w \to -1$ implies $\dot{\phi}^2/2V \ll 1$! Then

$$w = \frac{-3\epsilon - 1}{3\epsilon + 1} = -(1 - 3\epsilon)^2 + \mathcal{O}\left(\epsilon^2\right) \approx -1 + \frac{2}{3}\epsilon \tag{10.27}$$

where

$$\epsilon \equiv \frac{3\dot{\phi}^2}{2V}, \qquad \text{slow roll parameter} \tag{10.28}$$

So $w \approx -1$ implies that

$$\epsilon \ll 1 \Rightarrow \frac{\dot{\phi}^2}{2} \ll V$$

Another slow roll requirement: not only should $\frac{\dot{\phi}^2}{2}$ be smaller than V, but it should STAY smaller for a long time!

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V}{\partial \phi} = 0 \tag{10.29}$$

Damped harmonic oscillator! Should have the restoring force effects negligible compared to the "friction" term:

$$\ddot{\phi} \ll 3H\dot{\phi} \tag{10.30}$$

Define

$$\eta \equiv -\frac{\ddot{\phi}}{H\dot{\phi}}, \qquad \text{slow roll parameter} \tag{10.31}$$

require also that $\eta \ll 1$.

So: take ANY potential $V(\phi)$. As long as there exists some value of ϕ such that the potential is consistent with the conditions

$$\epsilon, \eta \ll 1 \tag{10.32}$$

inflation will occur when we have "equipartition" in the initial conditions.

Inflation HIGHLY REDUCES the amount of fine tuning of the initial conditions – requires that the initial value of ϕ is in the favourable region.

When ϵ, $\eta \ll 1$, the field equations for the zero mode reduce to

$$3H^2 = \frac{V}{M_p^2} \tag{10.33}$$

$$3H\dot{\phi} + \frac{\partial V}{\partial \phi} = 0 \tag{10.34}$$

Notice, that in general there would also be the spatial gradients; specifically:

$$g^{k\ell}\partial_k\phi\partial_\ell\phi \subset \rho \tag{10.35}$$

and

$$g^{k\ell}\partial_k\partial_\ell\phi \subset \nabla^2\phi = \Box\phi \tag{10.36}$$

Once inflation starts, however, it gets rid of anisotropies and inhomogeneities!

Notice that $\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2 \mathrm{d}\vec{x}^2$ has translational symmetry $\vec{x} \to \vec{x} + \vec{b}$ which allows us to Fourier transform the spatial coordinate dependence, and represent inhomogeneities as wave packets built out of plane waves

$$\phi_k = \phi_k(t) e^{i\vec{k}\cdot\vec{x}} \tag{10.37}$$

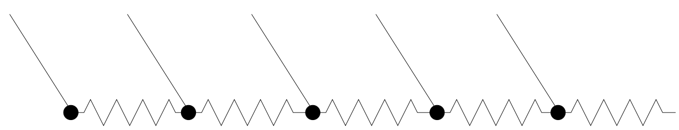
Then

$$g^{k\ell}\partial_k\phi\,\partial_\ell\phi = -\frac{k^2}{a^2}|\phi_k|^2 \tag{10.38}$$

$$g^{k\ell}\partial_k\partial_\ell\phi = -\frac{k^2}{a^2}\phi_k \tag{10.39}$$

and so as a blows up quickly these terms get wiped out!

The picture: QFT in an inflationary universe is like a system of linear harmonic oscillators frozen in the same place:



so only the zero mode survives!

Then from

$$3H^2 = \frac{V}{M_p^2} \tag{10.40}$$

$$3H\dot{\phi} = -\frac{\partial V}{\partial \phi} \tag{10.41}$$

solve for a, ϕ: note:

$$\frac{\dot{a}}{a} = \sqrt{\frac{V}{3M_p^2}} \tag{10.42}$$

$$\dot{\phi} = -\frac{\partial V/\partial\phi}{\sqrt{\frac{3V}{M_p^2}}} \tag{10.43}$$

so

$$\frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}\phi} = \frac{1}{a}\frac{\dot{a}}{\dot{\phi}} = -\frac{V}{M_p^2\,\frac{\partial V}{\partial \phi}} \tag{10.44}$$

Thus

$$\ln\frac{a}{a_0} = -\int_{\phi_0}^{\phi} \frac{V}{M_p^2\,\frac{\partial V}{\partial \phi}}\,\mathrm{d}\phi \tag{10.45}$$

and so

$$a = a_0 \exp\left(-\int_{\phi_0}^{\phi} \frac{V}{M_p^2\,\frac{\partial V}{\partial \phi}}\,\mathrm{d}\phi\right) \approx a_0 \exp\left(+\frac{V}{M_p^2\,|\frac{\partial V}{\partial \phi}|}\,\Delta\phi\right) \tag{10.46}$$

Bottom line: the slow roll conditions ϵ, $\eta \ll 1$ ensure that V, $\partial V/\partial\phi$ and the ratio of these terms are nearly constant. Then a small change in ϕ produces a HUGE increase in a yielding nearly exponential inflation!

Notice also that

$$a \approx a_0 \exp\left(\int_{t_0}^{t} \sqrt{\frac{V}{3M_p^2}}\,\mathrm{d}t\right) \approx a_0 \exp\left(\sqrt{\frac{V}{3M_p^2}}\,\Delta t\right) \tag{10.47}$$

$\therefore$ Inflation is approximated by de Sitter space metric in spatially flat slicing!

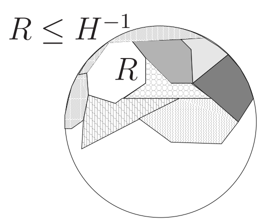
Chapter 11

Lecture 11

11.1 Geometry of inflation

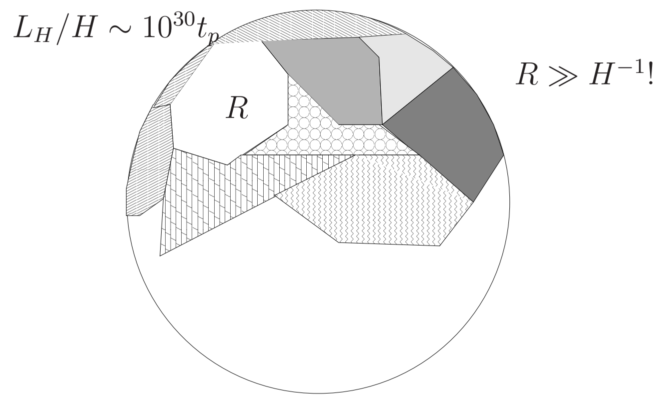
Pictorial representation of how inflation solves flatness, age, horizon and homogeneity problems:

Consider a closed, $k = 1$ inhomogenous universe:



where each patch is approximately homogenous and size H^{-1}.

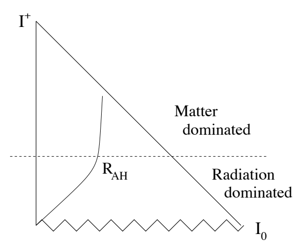
Blow up the balloon really fast:



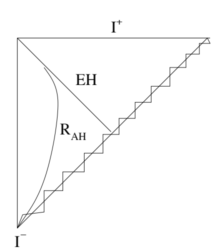
All of our universe fits into small homogenous patch!

11.1.1 Penrose diagram:

Recall for decelerating FRW spatially flat universe

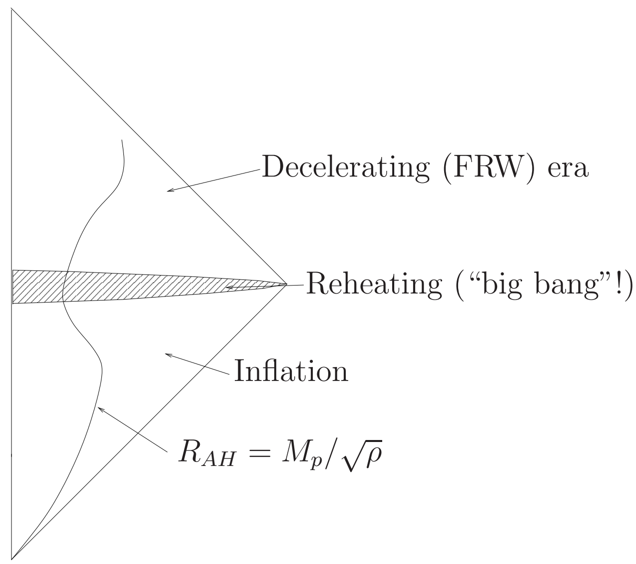


For inflating FRW universe with $k = 0$



what does the solution look like for an inflationary universe where inflation terminates and reheating occurs such that after inflation the universe is dominated by normal matter (relativistic or non-relativistic)?

Cut and paste: imagine an observer made of some indestructible material, who survives through inflation and reheating. The causal experiences of such an observer can be encoded in the causal patch:



Possibilities: $k = 0$, -1 or $+1$.

Inflation *does not* remove cosmological exotica – it postpones the epoch when they are relevant: indeed in the case of spatially closed universes, $k = 1$, if there is no cosmological constant in the future will recollapse WILL eventually occur, but it will take a long time for this to happen.

11.2 Role of the apparent horizon

Apparent horizon controls the growth of perturbations – i.e. inhomogeneities.

As a test consider a massless scalar field in an inflating patch:

$$\Box\phi = 0 \tag{11.1}$$

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2\mathrm{d}\vec{x}^2 \tag{11.2}$$

Consider the case when inflation is well-approximated by the de Sitter metric:

$$a = a_0 e^{H_0 t} \tag{11.3}$$

Then

$$\ddot{\phi} - \frac{1}{a^2}\vec{\nabla}^2\phi + 3H\dot{\phi} = 0 \tag{11.4}$$

For simplicity pick $a_0 = 1$ and change coordinates to conformal time:

$$\begin{aligned} \mathrm{d}s^2 &= -\mathrm{d}t^2 + e^{2H_0 t}\mathrm{d}\vec{x}^2 &(11.5)\\ &= e^{2H_0 t}\left(-(e^{-H_0 t}\mathrm{d}t) + \mathrm{d}\vec{x}^2\right) &(11.6)\\ \eta &= \int e^{-H_0 t}\,\mathrm{d}t = -\frac{1}{H_0}e^{-H_0 t} &(11.7)\\ \therefore \quad \mathrm{d}s^2 &= \frac{1}{H_0^2\eta^2}\left(-\mathrm{d}\eta^2 + \mathrm{d}\vec{x}^2\right) &(11.8) \end{aligned}$$

Note: $\eta < 0$ and flows from $-\infty$ to 0 as t goes from $-\infty$ to ∞.

$\eta \to 0$ limit of inflation!

Then

$$\begin{aligned} \dot{\phi} &= \phi'\frac{\mathrm{d}\eta}{\mathrm{d}t} = -H_0\eta\phi' &(11.9)\\ \ddot{\phi} &= H_0^2\eta(\eta\phi')' = H_0^2(\eta^2\phi'' + \eta\phi') &(11.10) \end{aligned}$$

So with this, $a = -(H_0\eta)^{-1}$ we have

$$\cancel{H_0^2}\eta^2\phi'' + \cancel{H_0^2}\eta\phi' - 3\cancel{H_0^2}\eta\phi' - \cancel{H_0^2}\eta^2\vec{\nabla}^2\phi = 0 \tag{11.11}$$

so

$$\eta^2\phi'' - 2\eta\phi' - \eta^2\vec{\nabla}^2\phi = 0 \tag{11.12}$$

First, decompose into Fourier modes

$$\phi = \phi_k(\eta)e^{i\vec{k}\cdot\vec{x}} \tag{11.13}$$

and second, transform $\phi_k(\eta) = \eta\varphi(\eta)$.

$$\begin{aligned} \therefore \phi_k &= \eta\varphi_k + \varphi_k &(11.14)\\ \phi'' &= \eta\varphi'' + 2\varphi'. &(11.15) \end{aligned}$$

So

$$\eta^3\varphi_k'' + \cancel{2\eta^2\varphi_k'} - \cancel{2\eta^2\varphi_k'} - 2\eta\varphi_k + \eta^3\vec{k}^2\varphi_k = 0 \tag{11.16}$$

or:

$$\varphi'' + \left(k - \frac{2}{\eta^2}\right)\varphi_k = 0 \tag{11.17}$$

where $k = |\vec{k}|$. This equation can be solved easily in the asymptotic regimes $k \gg \sqrt{2}/|\eta|$ and $k \ll \sqrt{2}/|\eta|$

$$\varphi_k = \begin{cases} A\cos(k\eta + \delta) & k \gg \frac{\sqrt{2}}{|\eta|} \\ \frac{\bar{A}}{\eta} + \bar{B}\eta^2 & k \ll \frac{\sqrt{2}}{|\eta|} \end{cases} \tag{11.18}$$

This yields, useing $\phi_k = \eta\varphi_k$:

$$\varphi_k = \begin{cases} A\eta\cos(k\eta + \delta) & k \gg \frac{\sqrt{2}}{|\eta|} \\ \bar{A} + \bar{B}\eta^3 & k \ll \frac{\sqrt{2}}{|\eta|} \end{cases} \tag{11.19}$$

or, noting that $a = -(H_0\eta)^{-1}$ and defining

$$\alpha = -\frac{A}{H_0}, \qquad \bar{\alpha} = \bar{A}, \qquad \bar{\beta} = -\frac{\bar{B}}{H_0^3} \tag{11.20}$$

we have

$$\varphi_k = \begin{cases} \frac{\alpha}{a}\eta\cos(k\eta + \delta) & k \gg \frac{\sqrt{2}}{|\eta|} \\ \bar{\alpha} + \frac{\bar{\beta}}{a^3} & k \ll \frac{\sqrt{2}}{|\eta|} \end{cases} \tag{11.21}$$

Note now: the physical wavelength at the Fourier mode with a fixed comoving wave number k is, from

$$e^{i\vec{k}\cdot\vec{x}} \to \vec{k}\cdot\vec{x} = \left(\frac{\vec{k}}{a}\right)(a\vec{x}) \tag{11.22}$$

$$\Rightarrow \vec{p} = \frac{\vec{k}}{a} \Rightarrow \lambda = \frac{a}{k} = -\frac{1}{H_0\eta k} \tag{11.23}$$

Then when $k \gg \sqrt{2}/|\eta|$, $\lambda \ll \frac{1}{\sqrt{2}H_0}$
and when $k \ll \sqrt{2}/|\eta|$, $\lambda \gg \frac{1}{\sqrt{2}H_0}$.

Apparent horizon: $R_{\rm AH} = 1/H_0$, so

$$k \gg \frac{\sqrt{2}}{|\eta|} \to \lambda \ll \frac{R_{\rm AH}}{\sqrt{2}} \tag{11.24}$$

$$k \ll \frac{\sqrt{2}}{|\eta|} \to \lambda \gg \frac{R_{\rm AH}}{\sqrt{2}} \tag{11.25}$$

$$\tag{11.26}$$

Thus: note that when

$$\lambda \ll \frac{R_{\rm AH}}{\sqrt{2}} \to \phi_k \to \frac{\alpha}{a}\cos(k\eta + \delta) \tag{11.27}$$

$$\lambda \gg \frac{R_{\rm AH}}{\sqrt{2}} \to \phi_k \to \bar{\alpha} + \frac{\bar{\beta}}{a^3} \tag{11.28}$$

FREEZE-OUT!!!!

Consider a mode with fixed $\vec{k}$. Say something in the early universe excited it; so there is some non-vanishing ϕ_k associated with it.

The *physical* wavelength is

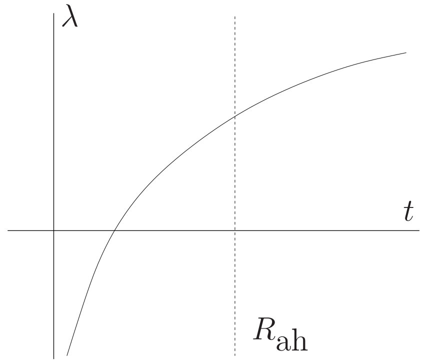
$$\lambda = \frac{a}{k} = -\frac{1}{kH_0\eta} = \frac{e^{Ht}}{k} \tag{11.29}$$

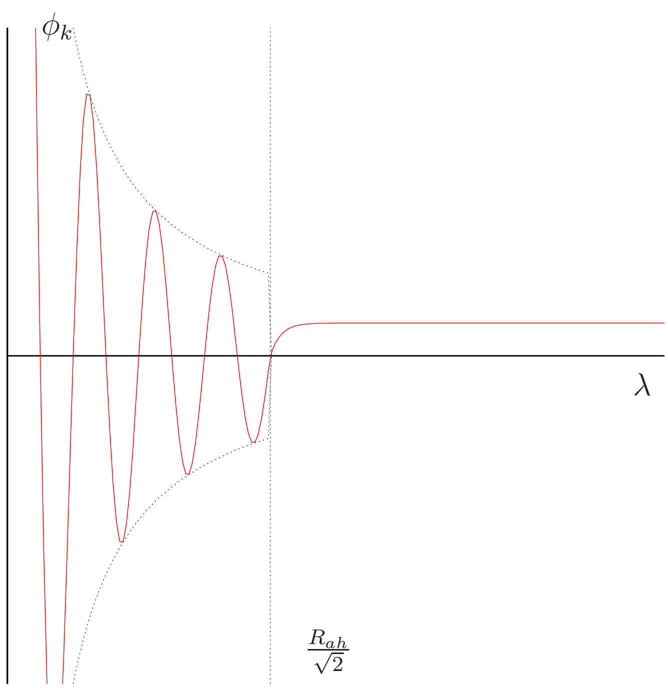
and as inflation proceeds ($t \to \infty$, $\eta \to 0^-$) it is getting exponentially stretched! While $\lambda < R_{\rm AH}/\sqrt{2}$, the mode behaves as a linear harmonic oscillator in a box with expanding walls:

$$\phi_k = \underbrace{\frac{\alpha}{a}}_{\text{redshift!}} \cos(k\eta + \delta) \tag{11.30}$$

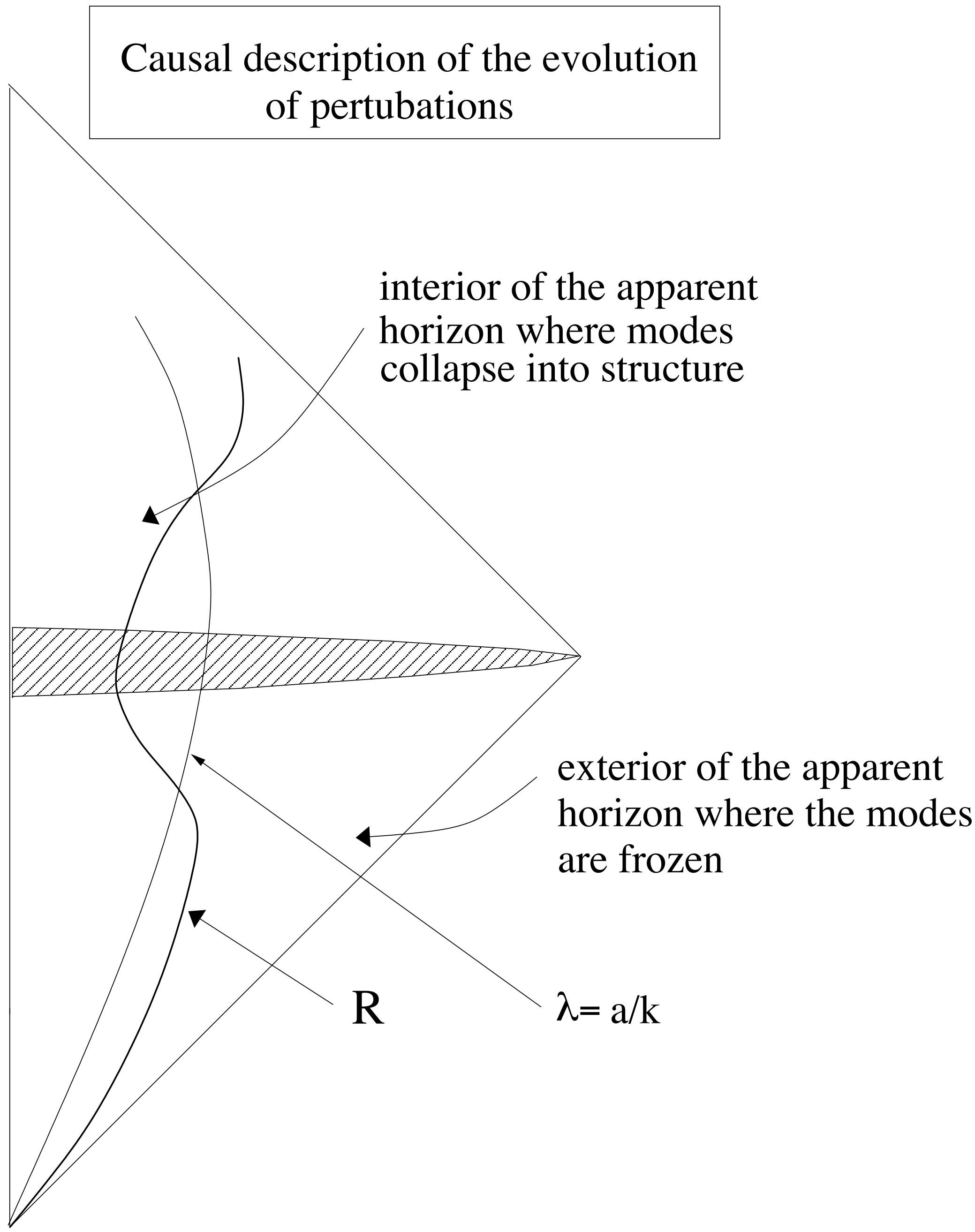
But once inflation stretches λ to scales greater than $R_{\rm AH}/\sqrt{2}$, the mode freezes:

$$\phi_k = \bar{\alpha} + \frac{\bar{\beta}}{a^3} \to \bar{\alpha}! \tag{11.31}$$


λ
t
R_{ah}


ϕ_k
λ
$\frac{R_{ah}}{\sqrt{2}}$

Perturbations with wavelengths greater than the apparent horizon leave a small imprint in the form of $\bar{\alpha}$.



Chapter 12

Lecture 12

12.1 QFT in FRW

Consider a Gaussian scalar field in an FRW background. The action is

$$S = -\int \mathrm{d}^4x\sqrt{g}\left(\frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2\right) \tag{12.1}$$

and on the fixed FRW background, ignoring backreaction (i.e. assuming $||T_{\mu\nu}^{(\phi)}|| \ll ||T_{\mu\nu}||$)

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2\mathrm{d}\vec{x}^2 \tag{12.2}$$

The field equation $(\Box - m^2)\phi = 0$ becomes

$$\ddot{\phi} + 3H\dot{\phi} - \frac{1}{a^2}\vec{\nabla}^2\phi + m^2\phi = 0 \tag{12.3}$$

Note now that $H = \dot{a}/a \neq$ const. Instead it is some function of t.

The right approach for solving this differential equation is to first decompose its solutions as plane waves in $\vec{x}$-space, which are the good basis functions consistent with the spatial Euclidean group. So

$$\phi(x) = \phi_k(t)e^{i\vec{k}\cdot\vec{x}} \tag{12.4}$$

and then, with $k = |\vec{k}|$,

$$\ddot{\phi}_k + 3H\dot{\phi}_k + \left(\frac{k^2}{a^2} + m^2\right)\phi_k = 0 \tag{12.5}$$

next, transform to the conformal time coordinate

$$\eta = \int \frac{\mathrm{d}t}{a}, \qquad \frac{\mathrm{d}\eta}{\mathrm{d}t} = \frac{1}{a} \tag{12.6}$$

So, let

$$\mathcal{H} = \frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}\eta} = \frac{a'}{a} \tag{12.7}$$

Thus

$$H = \frac{\dot{a}}{a} = \frac{a'}{a}\frac{\mathrm{d}\eta}{\mathrm{d}t} = \frac{\mathcal{H}}{a} \tag{12.8}$$

$$\dot{\phi}_k = \frac{\phi_k'}{a} \tag{12.9}$$

$$\ddot{\phi}_k = \frac{1}{a}\left(\frac{\phi_k'}{a}\right) = \frac{\phi_k''}{a^2} - \frac{\mathcal{H}\phi_k'}{a^2} \tag{12.10}$$

Then substituting

$$\phi_k'' + 2\mathcal{H}\phi_k' + (k^2 + m^2a^2)\phi_k = 0 \tag{12.11}$$

Now: get rig of the $\propto \mathcal{H}\phi_k'$ term. To see how, plug in

$$\phi_k = a^\alpha \varphi_k \tag{12.12}$$

Rewrite all the derivatives in terms of the new variable φ_k:

$$\phi_k' = a^\alpha \varphi_k' + \alpha a^\alpha \mathcal{H}\varphi_k = a^\alpha(\varphi_k' + \alpha\mathcal{H}\varphi_k) \tag{12.13}$$

$$\phi_k'' = a^\alpha(\varphi_k'' + 2\alpha\mathcal{H}\varphi_k' + \alpha(\mathcal{H}' + \alpha\mathcal{H}^2)\varphi_k) \tag{12.14}$$

Pick $\alpha = -1$! Then

$$\varphi_k'' - \cancel{2\mathcal{H}\varphi_k'} - (\mathcal{H}' - \cancel{\mathcal{H}^2})\varphi_k + \cancel{2\mathcal{H}\varphi_k'} - \cancel{2}\mathcal{H}^2\varphi_k + (k^2 + m^2a^2)\varphi_k = 0 \tag{12.15}$$

So

$$\varphi_k'' + (k^2 + m^2a^2 - \mathcal{H}' - \mathcal{H}^2)\varphi_k = 0 \tag{12.16}$$

Note that

$$\mathcal{H}' + \mathcal{H}^2 = \frac{a''}{a} \tag{12.17}$$

So the mode equation in the canonical form is

$$\varphi_k'' + \left(k^2 + m^2a^2 - \frac{a''}{a}\right)\varphi_k = 0 \tag{12.18}$$

For example: in spatially flat slicing in de Sitter,

$$a = e^{H_0 t} = -\frac{1}{H_0 \eta} \tag{12.19}$$

Hence

$$a'' = -\frac{2}{H_0 \eta^3}, \qquad \frac{a''}{a} = \frac{2}{\eta^2} \tag{12.20}$$

So

$$\varphi_k'' + \left(k^2 + \frac{m^2}{H_0^2 \eta^2} - \frac{2}{\eta^2}\right)\varphi_k = 0 \tag{12.21}$$

i.e.

$$\varphi_k'' + \left(k^2 - \frac{2 - \frac{m^2}{H_0^2}}{\eta^2}\right)\varphi_k = 0 \tag{12.22}$$

What about the exact mode solutions to these equations?

So: suppose that the universe is dominated by a single-component fluid with $p = w\rho$, then (lecture 6)

$$a = a_0 \left(\frac{t}{t_0}\right)^{\frac{2}{3(1+w)}} \tag{12.23}$$

$$\eta = \int t^{-\frac{2}{3(1+w)}}\,\mathrm{d}t = \begin{cases} \frac{3(1+w)}{3w+1} t^{\frac{3w+1}{3(w+1)}} & w \neq -\frac{1}{3} \\ \ln \frac{t}{t_0} & w = -\frac{1}{3} \end{cases} \tag{12.24}$$

So

$$a = \alpha_0 \left[(3w+1)\eta\right]^{\frac{2}{3w+1}} \tag{12.25}$$

$$\begin{aligned} w < -\frac{1}{3} \quad &\Rightarrow \quad \eta \in (-\infty, 0) \\ w > -\frac{1}{3} \quad &\Rightarrow \quad \eta \in (0, \infty) \\ w = -\frac{1}{3} \quad &\Rightarrow \quad \eta \in (-\infty, \infty) \end{aligned}$$

Notice also that

$$\frac{a''}{a} = \frac{2}{3w+1}\left[\frac{2}{3w+1} - 1\right]\frac{1}{\eta^2} \tag{12.26}$$

$$= \frac{2(1-3w)}{(1+3w)^2}\frac{1}{\eta^2} \tag{12.27}$$

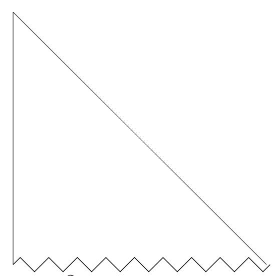
Therefore:

$$\varphi_k'' + \left(k^2 + m^2 \alpha_0^2 \left[(3w+1)\eta \right]^{\frac{4}{3w+1}} - \frac{2(1-3w)}{(1+3w)^2} \frac{1}{\eta^2} \right) \varphi_k = 0 \qquad (12.28)$$

Now consider the cases $w < -1/3$ and $w > -1/3$.

12.1.1 $w > -1/3$

$$+1 \leq \frac{4}{3w+1} < \infty$$
$$\eta \in (0, \infty)$$



Near the singularity the leading term is $1/\eta^2$; all the solutions, irrespective of the mass and the momentum, have universal behaviour which is VERY sensitive to the boundary conditions on the singularity.

$$\varphi_k \sim \eta^\alpha, \qquad \alpha = \frac{1}{2} \left(1 \pm \sqrt{1 + \frac{8(1-3w)}{3w+1)^2}} \right) \qquad (12.29)$$

These are some complicated contributions of plane waves of a fixed frequency, $e^{\pm i \omega \eta}$. Dressed up by strong interactions!!!!

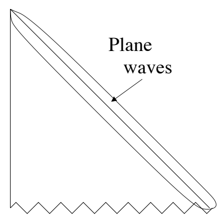
In the limit $\eta \to \infty$, the dominant term when $m \neq 0$ is the mass term (for fixed k).

It simply means that as the universe expands, the wavelength of the fluctuation increases and so the momentum contribution to the total frequency becomes negligible compared to the contribution coming from the mass term.

$\rightarrow$ the field becomes very NONRELATIVISTIC!

Let us ignore the mass term for now. Then, in this limit there is a bunch of LHOs!

$$\varphi_k'' + k^2 \varphi_k = 0 \tag{12.30}$$



12.1.2 $w < -1/3$

Now we have

$$-\infty < \frac{4}{3w+1} \leq -2, \qquad \eta \in (-\infty, 0) \tag{12.31}$$

The equation for a scalar field

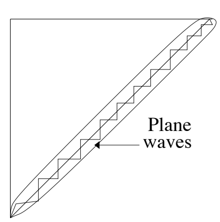
$$\varphi_k'' + \left(k^2 - \frac{2(1-3w)}{(1+3w)^2} \frac{1}{\eta^2} \right) \varphi_k = 0 \tag{12.32}$$

Note:

1)

In the limit $\eta \to -\infty$ both the terms proportional $\propto a''/a$ and the mass terms (which we are omitting) are negligible. Free field theory:

$$\to \varphi_k'' + k^2 \varphi_k = 0 \qquad \text{(LHOs)} \tag{12.33}$$



Bottomline:
Can pick the vacuum on null boundaries which admit solutions that behave like Minkowski space vacua – same short distance physics!

In the case $w < -1/3$ the null singularity is much more harmless than the spacelike singularity for the $w > -1/3$.

2)

When $\eta \to 0^-$, we have the freezeout phenomenon which we have discussed earlier for $m = 0$. For general m we can have instabilities!

Chapter 13

Lecture 13

13.1 QFT in FRW take II

Consider a massless scalar field

$$S = -\int \mathrm{d}^4x\sqrt{g}\frac{1}{2}(\partial\phi)^2 \tag{13.1}$$

In flat FRW $\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2\mathrm{d}\vec{x}^2$. The field equation

$$\ddot{\phi} + 3H\dot{\phi} - \frac{1}{a}\vec{\nabla}^2\phi = 0 \tag{13.2}$$

can be simplified by using the conformal time $\eta = \int \mathrm{d}t/a(t)$ and the substitution

$$\phi = \frac{1}{a}\varphi_k(\eta)e^{\pm i\vec{k}\cdot\vec{x}}. \tag{13.3}$$

The simplified equation of motion is

$$\varphi_k'' + \left(k^2 - \frac{a''}{a}\right)\varphi_k = 0 \tag{13.4}$$

We have also seen when $w \neq -1/3$ that

$$\frac{a''}{a} = \frac{2(1-3w)}{(1+3w)^2}\frac{1}{\eta^2} \tag{13.5}$$

Notice that this seems to blow up in the limit $w \to -1/3$ (cosmic strings). This is a slightly subtle case, since in this limit the coordinate transformation

$$\eta = \frac{3(1+w)}{1+3w} t^{\frac{3w+1}{3(1+w)}} \to \frac{-2}{0} t^0 \tag{13.6}$$

is ill-defined. (It blows up *and* all values of t map to one value of η!) We can fix it by L'Hopital's theorem to find that

$$\eta = a_0 \exp(\eta/\eta_0) \tag{13.7}$$

and so

$$\frac{a''}{a} = \frac{1}{\eta_0^2} \tag{13.8}$$

General mode functions can now be found; note that for $w = -1/3$ the modes are harmonic oscillators with a tachyonic mass $m^2 = 1/\eta_0^2$. *This is the instability of the static Einstein universe alluded to earlier!*

When $w \neq -1/3$

$$\varphi_k'' + \left(k^2 - \frac{\mu^2}{\eta^2}\right) \varphi_k = 0 \tag{13.9}$$

$$\mu^2 \equiv \frac{2(1-3w)}{(3w+1)^2} \qquad w \neq -\frac{1}{3} \tag{13.10}$$

What are the solutions to this equations? Rewrite as

$$\eta^2 \varphi_k'' + (\eta^2 k^2 - \mu^2)\varphi_k = 0 \tag{13.11}$$

and define the variable $\tau = k\eta$. Note that

$$\eta^2 \varphi_k'' = \tau^2 \frac{\mathrm{d}^2 \varphi_k}{\mathrm{d}\tau^2} \tag{13.12}$$

Bessel equation in disguise!

Indeed, define

$$\varphi_k = \sqrt{\tau}\psi_k \tag{13.13}$$

Then

$$\frac{\mathrm{d}\varphi_k}{\mathrm{d}\tau} = \sqrt{\tau}\frac{\mathrm{d}\phi_k}{\mathrm{d}\tau} + \frac{1}{2\sqrt{\tau}}\psi_k \tag{13.14}$$

$$\frac{\mathrm{d}^2\varphi_k}{\mathrm{d}\tau^2} = \sqrt{\tau}\frac{\mathrm{d}^2\psi_k}{\mathrm{d}\tau^2} + \frac{1}{\sqrt{\tau}}\frac{\mathrm{d}\psi_k}{\mathrm{d}\tau} - \frac{1}{4\tau^{3/2}}\psi_k \tag{13.15}$$

By substituting back we get

$$\tau^2\frac{\mathrm{d}^2\psi_k}{\mathrm{d}\tau^2} + \tau\frac{\mathrm{d}\psi_k}{\mathrm{d}\tau} + \left(\tau^2 - \mu^2 - \frac{1}{4}\right)\psi_k = 0 \tag{13.16}$$

To put this in the form of the canonical Bessel equation we define $\nu^2 = \mu^2 + 1/4$ and get

$$\tau^2\frac{\mathrm{d}^2\psi_k}{\mathrm{d}\tau^2} + \tau\frac{\mathrm{d}\psi_k}{\mathrm{d}\tau} + \left(\tau^2 - \nu^2\right)\psi_k = 0 \tag{13.17}$$

While the solutions are (linear combinations of) Bessel functions, it is convenient to use the particular linear combinations known as the *Hankel functions*, as these are the combinations that behave asymptotically as plane waves

$$\psi_k(\tau) = H_\nu^\pm(\tau) \xrightarrow{\tau\gg 1} \sqrt{\frac{2}{\pi\tau}}\exp\left(\pm i\left[\tau - \frac{2\nu+1}{4}\pi\right]\right) \tag{13.18}$$

and where

$$H_\nu^-(\tau)^\star = H_\nu^+(\tau) \tag{13.19}$$

For large τ, the variable $\varphi_k = \sqrt{\tau}H_\nu^\pm$ behave as the positive and negative frequency solutions. Recalling that $\tau = k\eta$:

$$\varphi_k \xrightarrow{\eta\gg 1} \sqrt{\frac{2}{\pi}}\exp\left(\pm i\left[k\eta - \frac{2\nu+1}{4}\pi\right]\right) \tag{13.20}$$

Ignore the phase as we can always absorb it away!

Thus the general solution for ϕ are linear superpositions of the momentum modes:

$$\phi(t,\vec{x}) = \frac{\sqrt{k\eta}}{a} H_\nu^\pm(k\eta) e^{\pm i\vec{k}\cdot\vec{x}} \tag{13.21}$$

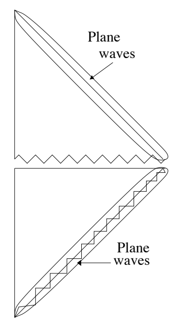
where

$$\nu = \sqrt{\mu^2 + \frac{1}{4}} = \sqrt{\frac{9(w-1)^2}{4(3w+1)^2}} = \frac{3}{2}\left|\frac{w-1}{1+3w}\right| \tag{13.22}$$

Note that

$$w > -1/3, \qquad \eta \in (0,\infty)$$
$$\text{so: } \tau \gg 1 \iff \eta \to \infty$$

$$w < -1/3, \qquad \eta \in (-\infty,0)$$
$$\text{so: } |\tau| \gg 1 \iff \eta \to -\infty$$



Recall the flat space field theory. There the solutions could be written as

$$\phi = \int \frac{d^3\vec{k}}{(2\pi)^3 2E_k}\left(a_{\vec{k}} e^{i(\vec{k}\vec{r}-E_k t)} + a_{\vec{k}}^\dagger e^{-i(\vec{k}\vec{r}-E_k t)}\right) \tag{13.23}$$

where

$$[a_{\vec{k}}, a_{\vec{q}}^\dagger] = (2\pi)^3 2E_k \delta^{(3)}(\vec{k}-\vec{q}) \tag{13.24}$$
$$\text{others} = 0 \tag{13.25}$$

and we can define the vacuum as the state annihilated by all $a_{\vec{k}}$:

$$a_{\vec{k}}|0>= 0 \tag{13.26}$$

Can we follow a similar procedure in this case, at least in the limit $\tau \gg 1$?

YES! we can devise the canonical quantisation procedure in FRW.

So start again with the action

$$S = -\int \mathrm{d}^4x\sqrt{g}\frac{1}{2}(\partial\phi)^2 \tag{13.27}$$

in $\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2\mathrm{d}\vec{x}^2$ coordinates:

$$S = \frac{1}{2}\int \mathrm{d}t\mathrm{d}^3\vec{x}\, a^3\left(\dot{\phi}^2 - \frac{1}{a^2}(\vec{\nabla}\phi)^2\right) \tag{13.28}$$

So the conjugate momenta are

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = a^3\dot{\phi} \tag{13.29}$$

and we can impose the equal-time canonical commutation relations

$$[\phi(t,\vec{x}), \pi(t,\vec{y})] = i\delta^{(3)}(\vec{x}-\vec{y}) \tag{13.30}$$

However: these variables are INCONVENIENT because of non-linearities!

Instead, we work with the conformal time coordinates and note $\mathrm{d}s^2 = a^2(-\mathrm{d}\eta^2 + \mathrm{d}\vec{x}^2)$

$$S = \frac{1}{2}\int \mathrm{d}\eta\,\mathrm{d}^3\vec{x}\,a^4\frac{\phi'^2 - (\vec{\nabla}\phi)^2}{a^2} \tag{13.31}$$

$$= \frac{1}{2}\int \mathrm{d}\eta\,\mathrm{d}^3\vec{x}\,a^2(\phi'^2 - (\vec{\nabla}\phi)^2) \tag{13.32}$$

Recalling $\varphi = a\phi$, we find that

$$a\phi' = (a\phi)' - a'\phi = (a\phi)' - a\mathcal{H}\phi = \varphi' - \mathcal{H}\varphi \tag{13.33}$$

Note: we CAN start with this action, upon noticing that the canonical momentum is

$$\pi = \frac{\partial\mathcal{L}}{\partial\varphi'} = \varphi' - \mathcal{H}\varphi \tag{13.34}$$

(similar to an electron in an external field!)

Alternatively:

$$(\varphi' - \mathcal{H}\varphi)^2 = \varphi'^2 + \mathcal{H}^2\varphi^2 - 2\mathcal{H}\varphi\varphi' \tag{13.35}$$

$$= \varphi'^2 + \mathcal{H}^2\varphi^2 - \mathcal{H}(\varphi^2)' \tag{13.36}$$

$$= \varphi'^2 + (\mathcal{H}^2 + \mathcal{H}')\varphi^2 - (\mathcal{H}\varphi^2)' \tag{13.37}$$

So

$$S = \frac{1}{2}\int \mathrm{d}\eta\, \mathrm{d}^3\vec{x} \left(\varphi'^2 + (\mathcal{H}^2 + \mathcal{H}')\varphi^2\right) - \underbrace{\frac{1}{2}\int \mathrm{d}\eta\, \mathrm{d}^3\vec{x}(\mathcal{H}\varphi^2)'}_{\text{boundary term}} \tag{13.38}$$

By dropping the boundary term (variations $\delta\varphi \to 0$ on the boundary) and recalling that

$$\mathcal{H}' + \mathcal{H}^2 = \frac{a''}{a} \tag{13.39}$$

we can put the action into its final form

$$S = \frac{1}{2}\int \mathrm{d}\eta\, \mathrm{d}^3\vec{x} \left(\varphi'^2 + \frac{a''}{a}\varphi^2\right) \tag{13.40}$$

Now:

$$P = \frac{\partial \mathcal{L}}{\partial \varphi'} = \varphi' \tag{13.41}$$

(we use P instead of the more cumbersome π_φ) and the equations of motion are

$$\varphi'' - \vec{\nabla}^2\varphi - \frac{a''}{a}\varphi = 0 \tag{13.42}$$

Chapter 14

Lecture 14

14.1 QFT in FRW take III

Canonical transformation: since

$$S = \frac{1}{2}\int \mathrm{d}\eta\, \mathrm{d}^3\vec{x}\left(\varphi'^2 - (\vec{\nabla}\varphi)^2 + \frac{a''}{a}\varphi^2\right) \tag{14.1}$$

we can follow the canonical quantisation procedure. Define the momentum

$$P = \frac{\partial \mathcal{L}}{\partial \varphi'} = \varphi' \tag{14.2}$$

and write down the canonical equal (conformal) time commutation relations

$$[\varphi(\eta,\vec{x}), P(\eta,\vec{y})] = i\delta^{(3)}(\vec{x}-\vec{y}) \tag{14.3}$$

$$[\varphi(\eta,\vec{x}), \varphi(\eta,\vec{y})] = [P(\eta,\vec{x}), P(\eta,\vec{y})] = 0 \tag{14.4}$$

From the field equation

$$\frac{\partial^2\varphi}{\partial\eta^2} + \frac{a''}{a}\varphi = 0 \tag{14.5}$$

we find that there exists a conserved current

$$j_\mu = i(\varphi^\star\partial_\mu\varphi - (\partial_\mu\varphi^\star)\varphi) \tag{14.6}$$

Thus the inner product on this space can be defined as usual

$$\langle\varphi_1|\varphi_2\rangle = i\int \mathrm{d}^3\vec{x}\left(\varphi_1^\star\varphi_2' - \varphi_1^{\star\prime}\varphi_2\right) \tag{14.7}$$

Note: in terms of an arbitrary coordinate system, and original, physical variable ϕ the inner product is[1]

$$\langle \phi_1 | \phi_2 \rangle = i \int \mathrm{d}^3\vec{x} \, \sqrt{g} g^{0\mu} \left(\phi_1^\star \nabla_\mu \phi_2 - (\nabla_\mu \phi_1^\star) \phi_2 \right) \tag{14.8}$$

We are interested in the Fourier transforms of φ since they give us a simple momentum space representation. So define

$$\varphi(\eta, \vec{x}) = \int \frac{\mathrm{d}^3 k}{(2\pi)^{3/2}} \left(b(\vec{k}, \eta) e^{i\vec{k}\cdot\vec{x}} + b^\dagger(\vec{k}, \eta) e^{-i\vec{k}\cdot\vec{x}} \right) \tag{14.9}$$

where we choose a slightly different normalisation because in curved space the concept of local energy density and is is tricker – just recall $\varphi'' - \vec{\nabla}^2 varphi - \frac{a''}{a}\varphi = 0$ where the "mass" is time-dependent.

From the definition of inner products, we can now deduce that the annihilation and creation operators are related to the Fourier transforms $\varphi_{\vec{k}}(\eta)$, $P_{\vec{k}}(\eta)$ of the fields and momenta

$$\varphi_{\vec{k}}(\eta) = \int \frac{\mathrm{d}^3\vec{x}}{(2\pi)^{3/2}} \varphi(\eta, \vec{x}) e^{-i\vec{k}\cdot\vec{x}} \tag{14.10}$$

$$P_{\vec{k}}(\eta) = \int \frac{\mathrm{d}^3\vec{x}}{(2\pi)^{3/2}} P(\eta, \vec{x}) e^{-i\vec{k}\cdot\vec{x}} \tag{14.11}$$

according to (adopting $k = |\vec{k}|$)

$$b(k, \eta) = \frac{1}{\sqrt{2}} \left(\sqrt{k} \varphi_k(\eta) + \frac{i}{\sqrt{k}} P(\eta) \right) \tag{14.12}$$

$$b^\dagger(k, \eta) = \frac{1}{\sqrt{2}} \left(\sqrt{k} \varphi_k(\eta) - \frac{i}{\sqrt{k}} P(\eta) \right) \tag{14.13}$$

[1] The reason that we refer to those as the "physical variables" is that it is *their* stress-energy tensor that couples directly to gravity

Note: while $\varphi^\dagger(\eta, \vec{x}) \equiv \varphi(\eta, \vec{x})$ we have

$$\varphi^\dagger_{\vec{k}}(\eta) = \int \frac{d^3\vec{x}}{(2\pi)^{3/2}} \varphi^\dagger(\eta, \vec{x}) e^{+i\vec{k}\cdot\vec{x}} \tag{14.14}$$

$$= \int \frac{d^3\vec{x}}{(2\pi)^{3/2}} \varphi(\eta, \vec{x}) e^{+i\vec{k}\cdot\vec{x}} \tag{14.15}$$

$$= \varphi_{-k}(\eta) \neq \varphi_k(\eta) \tag{14.16}$$

Same for the momentum!

Now, for the Fourier transforms the field equation reduces to

$$\varphi''_k(\eta) + \left(k^2 - \frac{a''}{a}\right)\varphi_k(\eta) = 0 \tag{14.17}$$

and as we know the solutions are the Hankel functions, which we now normalise as

$$u_k(\eta) = -\sqrt{\frac{\pi}{2}\eta} H^-_\nu(k\eta) \tag{14.18}$$

$$\text{and } u^\star_k = -\sqrt{\frac{\pi}{2}\eta} H^+_\nu(k\eta) \tag{14.19}$$

where

$$\nu = \frac{3}{2}\left|\frac{w-1}{1+3w}\right| \tag{14.20}$$

Note: we have absorbed away a $\sqrt{k}$, by writing the measure of the Fourier transform as $\propto d^3\vec{k}$ rather than $\propto d^3k/|\vec{k}|$.

Thus, since $H^+_\nu = H^{-\star}_\nu$

$$\varphi_k(\eta) = b(k)u_k + b^\dagger(-k)u^\star_k \tag{14.21}$$

and so

$$\varphi(\eta, \vec{x}) = \int \frac{d^3\vec{k}}{(2\pi)^{3/2}} \varphi_{\vec{k}}(\eta) e^{i\vec{k}\cdot\vec{x}} \tag{14.22}$$

$$= \int \frac{d^3\vec{k}}{(2\pi)^{3/2}} \left(b(\vec{k}) u_k e^{i\vec{k}\cdot\vec{x}} + b^\dagger(-\vec{k}) u_k^\star e^{i\vec{k}\cdot\vec{x}} \right) \tag{14.23}$$

$$= \int \frac{d^3\vec{k}}{(2\pi)^{3/2}} \left(b(\vec{k}) u_k e^{i\vec{k}\cdot\vec{x}} + \underbrace{b^\dagger(\vec{k}) u_{-k}^\star e^{-i\vec{k}\cdot\vec{x}}}_{k\to -k} \right) \tag{14.24}$$

Now from the canonical commutation relations one can see that, by the relations of b, $b^\dagger$ with φ, P we have indeed

$$[b(\vec{k}, \eta), b^\dagger(\vec{q}, \eta)] = \delta^{(3)}(\vec{k} - \vec{q}) \tag{14.25}$$

$$[b(\vec{k}, \eta), b(\vec{q}, \eta)] = [b^\dagger(\vec{k}, \eta), b^\dagger(\vec{q}, \eta)] = 0 \tag{14.26}$$

i.e. precisely the second quantisation algebra!

Thus we can define ADIABATIC vacuum (or thermal, Bunch-Davis or Euclidean) vacuum as the state $|\tilde{0}\rangle$ annihilated by $b(\vec{k}, \eta)$:

$$b(\vec{k}, \eta)|\tilde{0}\rangle = 0 \tag{14.27}$$

Note: in flat space limit

$$b(\vec{k}, \eta) = b(\vec{k}) e^{-ik\eta} \tag{14.28}$$

and so $|\tilde{0}\rangle$ is constant up to a phase – all of its evolution is reduced to an irrelevant phase factor!

THIS IS NOT THE CASE IN CURVED SPACE, and in particular not in FRW

In general,

$$b(\vec{k}, \eta) \neq b(\vec{k}) u_k(\eta). \tag{14.29}$$

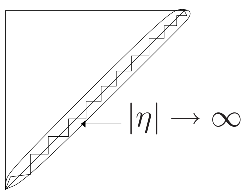
Howver, we know that

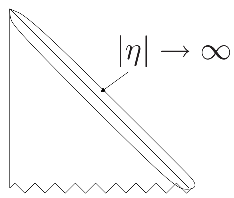
$$u_k(\eta) = -\sqrt{\frac{\pi}{2}}\eta H_\nu^-(k\eta) \tag{14.30}$$

$$\xrightarrow{|\eta|\to\infty} \frac{1}{\sqrt{2k}} e^{-ik\eta+i\delta} \tag{14.31}$$

and so INDEED in the limit $|\eta| \to \infty$, corresponding to the Minkowski like regimes

$w < -\frac{1}{3}$ $w > -\frac{1}{3}$





we do get

$$b(\vec{k}, \eta) \to \frac{1}{\sqrt{2k}} b(\vec{k}) e^{-ik\eta+i\delta} \tag{14.32}$$

and so this is WHY the adiabatic vacuum reduces (to leading order) to the usual Minkowski vacuum in this limit!

$$|\tilde{0}\rangle \xrightarrow{|\eta|\to\infty} |0\rangle \tag{14.33}$$

This is the technical reason why we pick this state.

However, as we will see

$$b(\vec{k}, \eta) = b(\vec{k}) f_k(\eta) + b^\dagger(-\vec{k}) g_k(\eta) \tag{14.34}$$

and so we get an admixture of a creation operator!

So

$$b(\vec{k}, \eta)|\tilde{0}\rangle = 0 \tag{14.35}$$

$$\to f_k(\eta) b(\vec{k})|\tilde{0}\rangle = -g_k(\eta) b^\dagger(-\vec{k})|\tilde{0}\rangle. \tag{14.36}$$

i.e. $|\tilde{0}\rangle$ is not *exactly* the same as the Minkowski vacuum, because the action of the annihilation operator corresponds with the opposite momentum (like a "hole"). This is called a SQUEEZED state (invented by Schrödinger in 1927).

But because

$$\frac{g_k(\eta)}{f_k(\eta)} \stackrel{|\eta|\to\infty}{\longrightarrow} 0 \tag{14.37}$$

(as we will soon see) this squeezed state reduces to the Minkowski vacuum.

In general: the formalism of transformations which mix b, $b^\dagger$ as

$$b(\vec{k},\eta) = b(\vec{k})f_k(\eta) + b^\dagger(-\vec{k})g_k(\eta) \tag{14.38}$$

$$b^\dagger(\vec{k},\eta) = b^\dagger(\vec{k})f_k^\star(\eta) + b(-\vec{k})g_k^\star(\eta) \tag{14.39}$$

is called a *Bogoliubov transformation*, as long as they preserve the canonical commutation relations. i.e.

$$[b(\vec{k},\eta), b^\dagger(\vec{q},\eta)] = i\delta(\vec{k}-\vec{q}) \iff [b(\vec{k}), b^\dagger(\vec{q})] = i\delta(\vec{k}-\vec{q}) \tag{14.40}$$

This requires

$$f_k^\star f_k - g_k^\star g_k = 1 \tag{14.41}$$

This means that the Bogoliubov transformations are unitary and thus fully consistent with quantum mechanics!

14.1.1 Particle production

Note how particle production comes about:

$$b(\vec{k},\eta)|\tilde{0}\rangle = 0 \tag{14.42}$$

$$\to b(\vec{k})|\tilde{0}\rangle = -\frac{g_k(\eta)}{f_k(\eta)} b^\dagger(-\vec{k})|\tilde{0}\rangle \tag{14.43}$$

so the number operator is $b^\dagger(\vec{k})b(\vec{k})$.

Then: the number of particles in the state $|\tilde{0}\rangle$ is

$$N = \langle\tilde{0}|b^\dagger(\vec{k})b(\vec{k})|\tilde{0}\rangle \tag{14.44}$$

Since

$$\langle\tilde{0}|b^\dagger(\vec{k}) = -\frac{g_k^\star}{f_k^\star}\langle\tilde{0}|b(-\vec{k}) \tag{14.45}$$

we have

$$N = \langle\tilde{0}|b^\dagger(\vec{k})b(\vec{k})|\tilde{0}\rangle \tag{14.46}$$

$$= \frac{|g_k|^2}{|f_k|^2}\langle\tilde{0}|b(-\vec{k})b^\dagger(-\vec{k})|\tilde{0}\rangle \tag{14.47}$$

$$= \frac{|g_k|^2}{|f_k|^2}(1+\ldots) \tag{14.48}$$

Thus the ratio $|g_k|^2/|f_k|^2$ measures the rate of particle production!! We will do this more accurately later on.

Chapter 15

Lecture 15

15.1 QFT in FRW take IV

So what are the solutions for $b(\vec{k},\eta)$, $b^\dagger(\vec{k},\eta)$? We could write down the explicit form of the Heisenberg equations of motion. Recalling

$$i\dot{Q} = [Q, H] \tag{15.1}$$

and solve them; but in fact we ALREADY know the solutions!!!

Recall

$$b(\vec{k},\eta) = \frac{1}{\sqrt{2}}\left(\sqrt{k}\varphi_{\vec{k}}(\eta) + \frac{i}{\sqrt{k}}P_{\vec{k}}(\eta)\right) \tag{15.2}$$

$$b^\dagger(\vec{k},\eta) = \frac{1}{\sqrt{2}}\left(\sqrt{k}\varphi^\dagger_{\vec{k}}(\eta) - \frac{i}{\sqrt{k}}P^\dagger_{\vec{k}}(\eta)\right) \tag{15.3}$$

$$P_{\vec{k}}(\eta) = \int \frac{\mathrm{d}^3x}{(2\pi)^{3/2}} P(\eta,\vec{x})e^{-i\vec{k}\cdot\vec{x}} \tag{15.4}$$

$$= \int \frac{\mathrm{d}^3x}{(2\pi)^{3/2}} \varphi'(\eta,\vec{x})e^{-i\vec{k}\cdot\vec{x}} \tag{15.5}$$

$$= \varphi'_k(\eta) \tag{15.6}$$

and

$$\varphi_{\vec{k}}(\eta) = b(\vec{k})u_k + b^\dagger(\vec{k})u^\star_k \tag{15.7}$$

where

$$u_k = -\sqrt{\frac{\pi}{2}\eta} H_\nu^-(k\eta) \tag{15.8}$$

$$u_k^\star = -\sqrt{\frac{\pi}{2}\eta} H_\nu^+(k\eta) \tag{15.9}$$

$$\text{with } \nu = \frac{3}{2}\left|\frac{w-1}{3w+1}\right| \tag{15.10}$$

Thus:

$$P_{\vec{k}}(\eta) = b(\vec{k})u_k' + b^\dagger(-\vec{k})u_k'^\star \tag{15.11}$$

and so

$$b(\vec{k},\eta) = \frac{1}{\sqrt{2}}\left(\sqrt{k}b(\vec{k})u_k + \sqrt{k}b^\dagger(-\vec{k})u_k^\star + \frac{i}{\sqrt{k}}b(\vec{k})u_k' + \frac{i}{\sqrt{k}}u_k'^\star\right) \tag{15.12}$$

$$= \frac{1}{\sqrt{2}}\left(\sqrt{k}u_k + \frac{i}{\sqrt{k}}u_k'\right)b(\vec{k}) + \frac{1}{\sqrt{2}}\left(\sqrt{k}u_k^\star + \frac{i}{\sqrt{k}}u_k'^\star\right)b^\dagger(-\vec{k}) \tag{15.13}$$

We can rewrite this as

$$b(\vec{k},\eta) = f_{\vec{k}}(\eta)b(\vec{k}) + g_{\vec{k}}(\eta)b^\dagger(-\vec{k}) \tag{15.14}$$

$$\text{where } f_{\vec{k}}(\eta) = \frac{1}{\sqrt{2}}\left(\sqrt{k}u_k + \frac{i}{\sqrt{k}}u_k'\right) \tag{15.15}$$

$$g_{\vec{k}}(\eta) = \frac{1}{\sqrt{2}}\left(\sqrt{k}u_k^\star + \frac{i}{\sqrt{k}}u_k'^\star\right) \tag{15.16}$$

Note: In the asymptotic limit $|\eta| \to \infty$,

$$u_k \to \frac{1}{\sqrt{2k}}e^{-ik\eta} \tag{15.17}$$

$$u_k^\star \to \frac{1}{\sqrt{2k}}e^{ik\eta} \tag{15.18}$$

so

$$f_k \to \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}e^{-ik\eta} + \frac{1}{\sqrt{2}}e^{-ik\eta}\right) = e^{-ik\eta} \tag{15.19}$$

$$g_k \to \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}e^{ik\eta} - \frac{1}{\sqrt{2}}e^{ik\eta}\right) = 0 \tag{15.20}$$

So indeed, as $|\eta| \to \infty$

$$b(\vec{k},\eta) \to b(\vec{k}) \tag{15.21}$$

i.e. Minkowski vacuum!

15.1.1 Bogoliubov rotation

We can rewrite the above relations in matrix form.

$$\begin{pmatrix} b(\vec{k},\eta) \\ b^\dagger(-\vec{k},\eta) \end{pmatrix} = \begin{pmatrix} f_{\vec{k}}(\eta) & g_{\vec{k}}(\eta) \\ g^\star_{\vec{k}}(\eta) & f^\star_{\vec{k}}(\eta) \end{pmatrix} \begin{pmatrix} b(\vec{k}) \\ b^\dagger(-\vec{k}) \end{pmatrix} \tag{15.22}$$

Thus we can we rewrite this, by inversion, as

$$\begin{pmatrix} b(\vec{k}) \\ b^\dagger(-\vec{k}) \end{pmatrix} = \begin{pmatrix} f^\star_{\vec{k}}(\eta) & -g_{\vec{k}}(\eta) \\ -g^\star_{\vec{k}}(\eta) & f_{\vec{k}}(\eta) \end{pmatrix} \begin{pmatrix} b(\vec{k},\eta) \\ b^\dagger(-\vec{k},\eta) \end{pmatrix} \tag{15.23}$$

Now, this is true for any η; so we have an evolution law for b, $b^\dagger$ from η_0 to η;

$$\begin{pmatrix} f^\star_{\vec{k}}(\eta_0) & -g_{\vec{k}}(\eta_0) \\ -g^\star_{\vec{k}}(\eta_0) & f_{\vec{k}}(\eta_0) \end{pmatrix} \begin{pmatrix} b(\vec{k},\eta_0) \\ b^\dagger(-\vec{k},\eta_0) \end{pmatrix} = \begin{pmatrix} f^\star_{\vec{k}}(\eta) & -g_{\vec{k}}(\eta) \\ -g^\star_{\vec{k}}(\eta) & f_{\vec{k}}(\eta) \end{pmatrix} \begin{pmatrix} b(\vec{k},\eta) \\ b^\dagger(-\vec{k},\eta) \end{pmatrix} \tag{15.24}$$

$$\therefore \begin{pmatrix} b(\vec{k},\eta) \\ b^\dagger(-\vec{k},\eta) \end{pmatrix} = \begin{pmatrix} f_{\vec{k}}(\eta) & g_{\vec{k}}(\eta) \\ g^\star_{\vec{k}}(\eta) & f^\star_{\vec{k}}(\eta) \end{pmatrix} \begin{pmatrix} f^\star_{\vec{k}}(\eta_0) & -g_{\vec{k}}(\eta_0) \\ -g^\star_{\vec{k}}(\eta_0) & f_{\vec{k}}(\eta_0) \end{pmatrix} \begin{pmatrix} b(\vec{k},\eta_0) \\ b^\dagger(-\vec{k},\eta_0) \end{pmatrix} \tag{15.25}$$

Thus

$$\begin{pmatrix} b(\vec{k},\eta) \\ b^\dagger(-\vec{k},\eta) \end{pmatrix} = \begin{pmatrix} u_{\vec{k}}(\eta,\eta_0) & v_{\vec{k}}(\eta,\eta_0) \\ v^\star_{\vec{k}}(\eta,\eta_0) & u^\star_{\vec{k}}(\eta,\eta_0) \end{pmatrix} \begin{pmatrix} b(\vec{k},\eta_0) \\ b^\dagger(-\vec{k},\eta_0) \end{pmatrix} \tag{15.26}$$

where

$$u_{\vec{k}} = f_{\vec{k}}(\eta) f^\star_{\vec{k}}(\eta_0) - g_{\vec{k}}(\eta) g^\star_{\vec{k}}(\eta_0) \tag{15.27}$$

$$v_{\vec{k}} = g_{\vec{k}}(\eta) f_{\vec{k}}(\eta_0) - f_{\vec{k}}(\eta) g_{\vec{k}}(\eta_0) \tag{15.28}$$

and by unitarity

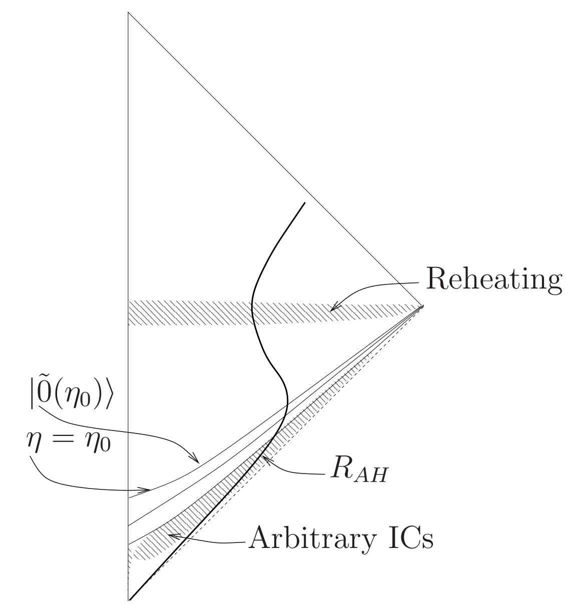
$$u^\star_k(\eta,\eta_0) u_k(\eta,\eta_0) - v^\star_k(\eta,\eta_0) v_k(\eta,\eta_0) = 1 \tag{15.29}$$

Thus finally we can rewrite $\varphi_{\vec{k}}(\eta)$ in terms of $b(\vec{k}, \eta_0$ and $b^\dagger(\vec{k}, \eta_0)$ rather than $b(\vec{k})$, $b^\dagger(\vec{k})$. We have

$$\varphi_{\vec{k}} = \frac{1}{\sqrt{2k}} \left[u_{\vec{k}}(\eta, \eta_0) + v^\star_{\vec{k}}(\eta, \eta_0) \right] b(\vec{k}, \eta_0) + \frac{1}{\sqrt{2k}} \left[u^\star_{\vec{k}}(\eta, \eta_0) + v_{\vec{k}}(\eta, \eta_0) \right] b^\dagger(-\vec{k}, \eta_0) \tag{15.30}$$

This is an important piece of mathematics!!!

The reason: consider again the space-time picture of inflation:



If we choose as the initial state the vacuum annihilated by $b(\vec{k})$

$$b(\vec{k})|0\rangle = 0 \tag{15.31}$$

we *must* push the η =const surface to $-\infty$ and *declare* that the state of the universe was the Minkowski vacuum then – but this is equivalent to the choice of special initial conditions!!!! This is contrary to the philosophy of inflation! INSTEAD we say that the initial state was an arbitrary, messy, non-vaccum state which was dissipated away by inflation! So

1. Start with an arbitrary initial surface

2. Get inflation to run for $\mathcal{O}(10)$ e-folds – it IRONS OUT the universe removing initial wrinkles. Recall that $\rho \sim w^{3(1+w)} \sim e^{-3(1+w)Ht}$.

3. The subsequent state of the universe is VERY close to the instantaneous vacuum at that time! So

$$b(\vec{k}, \eta_0)|\tilde{0}\rangle = 0 \tag{15.32}$$

and so $|\tilde{0}\rangle$ is the approximate state of the universe at the time η_0, some $\mathcal{O}(10)$ e-folds after the onset of inflation.

4. From that time on, the future evolution is completely encoded by the subsequent adiabatic evolution of $|\tilde{0}\rangle$ and the evolution of the quantum fields $\varphi_{\vec{k}}(\eta)$.

Fluctuations of fields are produced in the vacuum $|\tilde{0}\rangle$ as dictated by the standard short distance QFT; but because of cosmic expansion the waves with $\lambda = a/k$ get stretched out of the horizon and can end up frozen, i.e. the particles out of which these waves are composed fail to annihilate!

15.1.2 Two-point function

Two-point function $\delta\phi_k =?$. Measure of fluctuations.

$$\delta\phi_{\vec{k}} = ||\phi_k|| \tag{15.33}$$

$$||\phi_k||^2 \delta^{(3)}(\vec{k} - \vec{q}) = \frac{k^3}{2\pi^2} \langle \phi_{\vec{k}} \phi_{\vec{q}}^\dagger \rangle \tag{15.34}$$

Recall that $\phi = \varphi/a$; thus:

$$||\phi_k||^2 \delta^{(3)}(\vec{k} - \vec{q}) = \frac{k^3}{2\pi^2 a^2} \langle \varphi_{\vec{k}} \varphi_{\vec{q}}^\dagger \rangle \tag{15.35}$$

Let the state be $|\tilde{0}(\eta_0)\rangle$:

$$\langle \varphi_{\vec{k}} \varphi_{\vec{q}}^\dagger \rangle = \langle \tilde{0}(\eta_0) | \varphi_{\vec{k}} \varphi_{\vec{q}}^\dagger | \tilde{0}(\eta_0) \rangle \tag{15.36}$$

Substituting $\varphi_{\vec{k}}$ and using

$$b(\vec{k}, \eta_0) b^\dagger(\vec{q}\eta_0) |\tilde{0}(\eta_0)\rangle = \delta^{(3)}(\vec{k} - \vec{q}) |\tilde{0}\rangle \tag{15.37}$$

$$\text{others}|\tilde{0}\rangle = 0 \tag{15.38}$$

we get

$$\langle \varphi_{\vec{k}} \varphi^\dagger_{\vec{q}} \rangle = \delta^{(3)}(\vec{k} - \vec{q}) \frac{1}{2k} |u_{\vec{k}}(\eta, \eta_0) + v^\star_{\vec{k}}(\eta, \eta_0)|^2 \tag{15.39}$$

To get the relevant limit, we are interested in finding $\delta\phi_{\vec{k}}$ as $\eta \to 0$, as this is the portion that SURVIVES long inflation. Assuming de Sitter

$$a = e^{H_0 t} = -\frac{1}{H_0 t} \tag{15.40}$$

Then

$$u_k(\eta) = \frac{1}{\sqrt{2k}} \left(1 - \frac{i}{k\eta}\right) e^{-ik\eta} \tag{15.41}$$

$$u^\star_k(\eta) = \frac{1}{\sqrt{2k}} \left(1 + \frac{i}{k\eta}\right) e^{ik\eta} \tag{15.42}$$

So we have

$$U_{\vec{k}}(\eta, \eta_0) + V_{\vec{k}}(\eta, \eta_0) \xrightarrow[]{\substack{\eta \to 0 \\ \eta_0 \to -\infty}} -\frac{i}{k\eta} e^{-ik(\eta - \eta_0)} \tag{15.43}$$

So the fluctuations go as

$$(\delta\phi_k)^2 = \frac{k^3}{2\pi^2 a^2} \frac{|u_k(\eta, \eta_0) + v^\star_k(\eta, \eta_0)|^2}{2k} \tag{15.44}$$

$$= \frac{\cancel{k^3}}{2\pi^2} H_0^2 \cancel{\eta^2} \frac{1}{2\cancel{k^3}\cancel{\eta^2}} = \left(\frac{H_0}{2\pi}\right)^2 \tag{15.45}$$

$$\therefore \delta\phi_k = \frac{H_0}{2\pi} \tag{15.46}$$

Fluctuations of a relativistic field are set by the expansion rate of the universe!

$$T_H = \frac{H_0}{2\pi} \tag{15.47}$$

Hawking temperature of de Sitter space, see [10]. So this is the behaviour of a relativistic QFT in thermal equilibrium with some (thermal) heat bath.

Chapter 16

Lecture 16

16.1 Gauge invariant perturbation theory

So far we have discussed the evolution of perturbations in a fixed background, ignoring backreaction! [11]

But:

$$R^\mu{}_\nu - \frac{1}{2}\delta^\mu{}_\nu R = \frac{1}{M_p^2}T^\mu{}_\nu \tag{16.1}$$

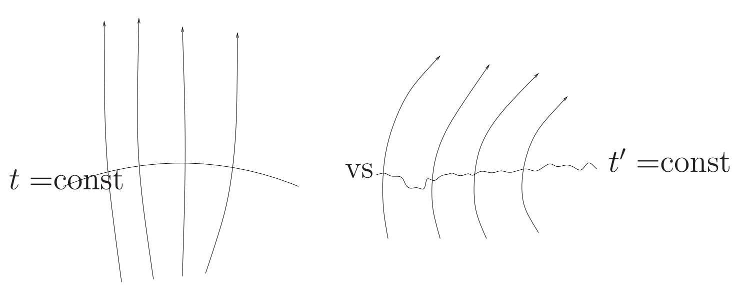
implies that matter perturbations source the metric perturbations!

$$\phi \to \phi + \delta\phi \Rightarrow T^\mu{}_\nu \to T^\mu{}_\nu + \delta T^\mu{}_\nu \tag{16.2}$$

$$\Rightarrow R^\mu{}_\nu \to R^\mu{}_\nu + \delta R^\mu{}_\nu \Rightarrow g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu} \tag{16.3}$$

16.1.1 The problem:

Some perturbations of the metric can be UNDONE with coordinate transformations!!!



How do we measure real, physical perturbations, and distinguish them from the pure gauge mode (that portion of the perturbation that can be undone by a coordinate transformation)?

The idea: recall that the coordinate transformations change the metric to

$$g_{\mu\nu}(x) \to \bar{g}_{\mu\nu}(\bar{x}) = g_{\alpha\beta}(x)\frac{\partial x^\alpha}{\partial \bar{x}^\mu}\frac{\partial x^\beta}{\partial \bar{x}^\nu} \tag{16.4}$$

We derive the effect of these transformations at the level of <u>linearised</u> theory!!!

Expectation: inflation smoothes out the large wrinkles in the fabric of space-time, but out of some small wrinkles many survive, as we have seen from $\delta\phi_{\vec{k}} \sim H_0$. So, we expect that they should be described as small perturbations on top of the smooth backgrounds; observations conform this, since at large scales

$$\frac{\delta\rho}{\rho} \sim 10^{-5} \tag{16.5}$$

During inflation

$$\begin{aligned} \phi &= \text{inflaton} \\ g_{\mu\nu} &= \text{background metric} \\ \therefore \phi &= \phi_0 + \delta\phi \\ g_{\mu\nu} &= g^0_{\mu\nu} = h_{\mu\nu} \end{aligned}$$

Take $g^0_{\mu\nu}$ to just be flat FRW as we have seen that this is a good approximation for inflation, after some initial period since the beginning of inflation.

So:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a_0^2\,\mathrm{d}\vec{x}^2 \tag{16.6}$$

$$3H_0^2 = \frac{1}{M_p^2}\left(\frac{\dot{\phi}_0^2}{2} + V\right) \tag{16.7}$$

$$\ddot{\phi} + 3H_0\dot{\phi} + \frac{\partial V}{\partial \phi_0} = 0 \tag{16.8}$$

and in slow roll, $\epsilon = 3\dot{\phi}_0^2/2V$, $\eta = -\ddot{\phi}H_0\dot{\phi} \ll 1$.

Consider the perturbations then; first rewrite everything in terms of the conformal time. Then split the perturbation tensor $h_{\mu\nu}$, which is the irreducible representation of the Lorentz group, into irreducible representations of the rotation group SO(3). The reason is that time-dependence breaks the symmetry.

$$\mathrm{d}s_4^2 = a^2\Big(-(1+2\Psi)\mathrm{d}\eta^2 + (1+2\Phi)\mathrm{d}\vec{x}^2 + (h_{kl} + \nabla_{(k}X_{l)} + \nabla_k\nabla_l E)\mathrm{d}x^k\,\mathrm{d}x^l + 2(\nabla_k B + V_k)\mathrm{d}x^k\mathrm{d}\eta\Big) \tag{16.9}$$

Here

- Ψ, Φ, E, B are scalars
- V_k, X_k are vectors
- h_{kl} is a tensor

h_{kl} is the true dynamical degree of gravity – this is the two graviton polarisations!
$\Rightarrow h_{kl}$ propagates on its own!

$V_k, X_k, \Psi, \Phi, E, B$: gauge degrees of freedom.
These fields are NON-PROPAGATING. In the absence of sources they can be gauged away!

During inflation: the source is the perturbation of the field $\to$ this means that to leading order NOTHING sources the two vectors!

Why? Recall that when we write out the equations of motion for the perturbation they are *linear* by our assumption:

$$L\psi_{\{a\}} = m\phi_{\{a\}} \tag{16.10}$$

indicies must coincide by rotational symmetry!

Because there are no vectors in the matter sector (any vector is getting rapidly inflated away, because vacuum energy pushes the system towards ISOTROPY, which breaks a background vector breaks, so the two are inconsistent). V_k and X_k do not get produced significantly during inflation (no production at the linear order, possible at the quadratic order though: for example $\partial_k \delta\phi h^k{}_l$).

PROBLEM WITH THE ORIGIN OF MAGNETIC FIELDS!

Sp we can separate out, in linear order perturbation theory, the dynamics of tensor, vector and scalar perturbations.

We will focus on the scalars, they are the distinct prediction of inflation!

So

$$\mathrm{d}s_4^2 = a^2 \left\{ -(1+2\Psi)\mathrm{d}\eta^2 + (1+2\Phi)\mathrm{d}\vec{x}^2 + \nabla_k \nabla_l E \, \mathrm{d}x^k \, \mathrm{d}x^l + 2\nabla_k B \, \mathrm{d}x^k \mathrm{d}\eta \right\}$$

Not all of these scalars are independent!

Note: make a gauge transformation

$$x^k \to \bar{x}^k = x^k + f(\eta) g^{kl} \nabla_l Q_m \tag{16.11}$$

$$\eta \to \bar{\eta} = \eta + g(\eta) Q_m \tag{16.12}$$

where $Q_m = e^{\pm i\vec{m}\cdot\vec{x}}$ – plane waves.
∴ appropriate wave expansion for spatially flat FRW.

Then

$$\mathrm{d}x^k \to \mathrm{d}\bar{x}^k = \mathrm{d}x^k + f_m' g^{kl} \nabla_l Q_m \, \mathrm{d}\eta + f_m g^{kl} \nabla_j \nabla_l Q_m \, \mathrm{d}x^j \tag{16.13}$$

$$\mathrm{d}\eta \to \mathrm{d}\bar{\eta} = \mathrm{d}\eta + g_m' Q_m \mathrm{d}\eta + g_m \nabla_k Q_m \, \mathrm{d}x^k \tag{16.14}$$

Now $\mathrm{d}s^2 = \mathrm{d}\bar{s}^2$, where

$$\mathrm{d}\bar{s}^2 = a^2 \left\{ -(1+2\bar{\Psi})\mathrm{d}\bar{\eta}^2 + (1+2\bar{\Phi})\mathrm{d}\vec{\bar{x}}^2 + \bar{\nabla}_k \bar{\nabla}_l \bar{E} \mathrm{d}\bar{x}^k \mathrm{d}\bar{x}^l + 2\bar{\nabla}_k \bar{B} \mathrm{d}\bar{x}^k \mathrm{d}\bar{\eta} \right\}$$

Write out $\mathrm{d}\bar{\eta}$, $\mathrm{d}\bar{x}^k$ explicitly, collect like terms, and keep only to linear order. Then compare to the original metric coefficents Ψ, E, B and Φ: (expanding $\Psi = \Psi_m Q_m$ etc)

$$\bar{\Psi}_m = \Psi_m - g_m' - \frac{a'}{a} g_m \tag{16.15}$$
$$\bar{\Phi}_m = \Phi_m - \frac{a'}{a} g_m \tag{16.16}$$
$$\bar{E}_m = E_m - 2 f_m \tag{16.17}$$
$$\bar{B}_m = B_m + g_m - f_m' \tag{16.18}$$

Also consider the effect of the gauge transformations on the matter (inflation) field:

$$\phi = \phi_0 + \delta\phi \tag{16.19}$$

Then: scalar transformation properties imply

$$\phi_0(\eta, x) + \delta\phi(\eta, x) = \phi_0(\bar{\eta}, \bar{\vec{x}}) + \delta\bar{\phi}(\bar{\eta}, \bar{\vec{x}}) \tag{16.20}$$

So

$$\delta\bar{\phi} = \delta\phi - [\phi_0(\bar{\eta}, \bar{x}) - \phi(\eta, x)] \tag{16.21}$$
$$= \delta\phi - \phi_0' g_m Q_m \tag{16.22}$$

We can rewrite this as

$$\delta\phi = \delta\phi_m Q_m \tag{16.23}$$

where, under a gauge transformation

$$\delta\phi_m \rightarrow \delta\bar{\phi}_m = \delta\phi_m - \phi_0' g_m \tag{16.24}$$

This is the linearised, infinitesimal form of the diffeomorphisms

$$g_{\mu\nu} \rightarrow \bar{g}_{\mu\nu} = g_{\alpha\beta} \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \frac{\partial x^\beta}{\partial \bar{x}^\nu} \tag{16.25}$$
$$\phi \rightarrow \bar{\phi}(\bar{x}) = \phi(x) \tag{16.26}$$

We can now construct the gauge invariant variables (drop m and use $\mathcal{H}_0 = a_0'/a_0$) $\rightarrow$ gauge invariant potentials

$$\hat{\Theta} = \Phi - \frac{\mathcal{H}_0}{\phi_0'}\delta\phi \tag{16.27}$$

$$\hat{\Phi} = \Phi + \mathcal{H}_0\left(B - \frac{E'}{2}\right) \tag{16.28}$$

$$\hat{\Psi} = \Psi + \mathcal{H}_0\left(B - \frac{E'}{2}\right) + B' - \frac{E''}{2} \tag{16.29}$$

Now we can pick a different gauge and work in it or we can just take general setup and once we get linearised equations of motion, re-express them in terms of the gauge invariant variables.

The latter approach is way too complicated, and not necessary. Just pick a gauge, work out the equations and identify the variables with (a linear combination of) gauge potentials.

Convenient gauge: LONGITUDINAL

$$E = B = 0 \tag{16.30}$$

Starting with some general coordinates, pick gauge transformations according to (16.11) and (16.12):

$$x^k \rightarrow \bar{x}^k = x^k + f(\eta)g^{kl}\nabla_l Q_m$$
$$\eta \rightarrow \bar{\eta} = \eta + g(\eta)Q_m$$

such that

$$f = \frac{E}{2}, \qquad \text{and } g = \frac{E'}{2} - B \tag{16.31}$$

This then guarantees that $\bar{E} = \bar{B} = 0$.

Ok, so then just drop the overbars, assuming that we started with these coordinates to begin with. In this gauge

$$\hat{\Psi} = \Psi \tag{16.32}$$

$$\hat{\Phi} = \Phi \tag{16.33}$$

so

$$\mathrm{d}s^2 = a^2 \left\{ -(1+2\hat{\Psi})\mathrm{d}\eta^2 + (1+2\hat{\Phi})\mathrm{d}\vec{x}^2 \right\} \tag{16.34}$$

where $\hat{\Psi}$ and $\hat{\Phi}$ are exactly gauge invariant variables $\rightarrow$ the equations will automatically be gauge invariant!

Chapter 17

Lecture 17

17.1 Gauge invariant scalar perturbations

After we have identified the gauge invariant potentials, we can proceed with getting the linearised perturbation equations for them.

We start with the field equations, where the stress-energy is set up by the inflation field:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{1}{M_p^2}\left(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}(\partial\phi)^2 - g_{\mu\nu}V(\phi)\right) \tag{17.1}$$

$$\nabla^2\phi = \frac{\partial V}{\partial\phi} \tag{17.2}$$

The equations for the FRW flat background dominated by the scalar zero mode in the slow roll regime are

$$3\mathcal{H}_0^2 = \frac{1}{M_p^2}\left(\frac{(\phi_0')^2}{2} + a_0^2 V(\phi_0)\right) \tag{17.3}$$

$$\phi_0'' + 2\mathcal{H}_0\phi_0' + a_0^2\frac{\partial V}{\partial\phi_0} = 0 \tag{17.4}$$

and the perturbations are

$$\mathrm{d}s^2 = a_0^2\left(-(1+2\Psi)\mathrm{d}\eta^2 + (1+2\Phi)\mathrm{d}\vec{x}^2\right) \tag{17.5}$$

$$\phi = \phi_0 + \delta\phi \tag{17.6}$$

So now the perturbations are (slightly!) simpler to compute if we rewrite the Einstein's equations by tracing using $R = -T/M_p^2$ as

$$R_{\mu\nu} = \frac{1}{M_p^2}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}t\right) \tag{17.7}$$

$$\text{i.e. } R_{\mu\nu} = \frac{1}{M-p^2}\tau_{\mu\nu} \tag{17.8}$$

Thus

$$\delta R_{\mu\nu} = \frac{1}{M_p^2}\delta\tau_{\mu\nu} \tag{17.9}$$

Now, in terms of $h_{\mu\nu}$

$$\delta R_{\mu\nu} = \frac{1}{2}\nabla_\lambda\nabla_\mu h_\nu{}^\lambda + \frac{1}{2}\nabla_\lambda\nabla_\nu h_\mu{}^\lambda - \frac{1}{2}\nabla^2 h_{\mu\nu} - \frac{1}{2}\nabla_\mu\nabla_\nu h \tag{17.10}$$

Now a systematic calculation shows, using also perturbations of $\tau_{\mu\nu}$:

0-0 equation :

$$\nabla^2\Psi = 3\Phi'' + 3\mathcal{H}_0(\Phi' - \Psi') + \frac{1}{M_p^2}\left(2\phi_0'\delta\phi - a_0^2\frac{\partial V}{\partial\phi_0}\delta\phi - 2a_0^2V\Psi\right) \tag{17.11}$$

0-k equation :

$$\Phi' = \mathcal{H}_0\Psi - \frac{1}{2M_p^2}\phi_0 \tag{17.12}$$

k-l equation :

$$\partial_k\partial_l(\Phi + \Psi) + \delta_{kl}\Big\{\vec{\nabla}^2\Phi - \Phi'' - 5\mathcal{H}_0\Phi' + \mathcal{H}_0\Psi' + 2\mathcal{H}_0^2(\Psi - \Phi) + 2\frac{a_0''}{a_0}(\Psi - \Phi)$$
$$+ \frac{1}{M_p^2}\left(2a_0^2V_0\Phi + a_0^2\frac{\partial V}{\partial\phi_0}\delta\phi\right)\Big\} = 0 \tag{17.13}$$

ϕ equation :

$$\delta\phi'' + 2\mathcal{H}_0\delta\phi' - \vec{\nabla}^2\delta\phi + a_0^2\frac{\partial^2 V}{\partial\phi_0^2}\delta\phi = 2\Psi(\phi_0'' + 2\mathcal{H}_0\phi_0') + (\Psi' - 3\Phi')\phi_0'$$

Now perform the spatial Fourier transform

$$f = f_k(\eta)e^{i\vec{k}\vec{x}} \tag{17.14}$$

So consider the k–l equation, it is of the form

$$-Mk_kk_l + \delta_{kl}N = 0 \tag{17.15}$$

Take the trace:

$$4N = Mk^2 \rightarrow N = \frac{Mk^2}{4} \tag{17.16}$$

Thus

$$M\left(k_kk_l - \frac{k^2}{4}\delta_{kl}\right) = 0 \tag{17.17}$$

must be valid for all momenta; contract with k_l:

$$0 = Mk^2\left(k_k - \frac{1}{4}k_k\right) = \frac{3}{4}Mk^2k_k \tag{17.18}$$

$$\Rightarrow M = N = 0 \tag{17.19}$$

Hence:

$$\Psi = -\Phi \tag{17.20}$$

The remaining field equations are

$$\Phi'' + 3\mathcal{H}_0\Phi' + (\mathcal{H}_0' + 2\mathcal{H}^2)\Phi + \frac{1}{2M_p^2}\left(\phi_0'\delta\phi' - a_0^2\frac{\partial V}{\partial\phi_0}\delta\phi\right) = 0 \tag{17.21}$$

$$-k^2\Phi - 3\mathcal{H}_0\Phi' - (\mathcal{H}_0' + 2\mathcal{H}_0^2)\Phi + \frac{1}{2M_p^2}\left(\phi_0'\delta\phi' + a_0^2\frac{\partial V}{\partial\phi_0}\delta\phi\right) = 0 \tag{17.22}$$

$$\Phi' + \mathcal{H}_0\Phi + \frac{1}{2M_p^2}\phi_0'\delta\phi = 0 \tag{17.23}$$

$$\delta\phi'' + 2\mathcal{H}_0\delta\phi' + a_0^2\frac{\partial^2 V}{\partial\phi_0^2}\delta\phi + k^2\delta\phi - 2\Phi a_0^2\frac{\partial V}{\partial\phi_0} + 4\phi_0'\Phi = 0 \tag{17.24}$$

Note that (17.23) allows us to solve for Φ in terms of ϕ (and, of course, one arbitrary integration constant).

This looks like a mess; however, one can show that the variable

$$\varphi = a_0\delta\phi - \frac{a_0\phi_0'}{\mathcal{H}_0}\Phi = -Z\Theta \tag{17.25}$$

where

$$Z \equiv \frac{a_0\phi_0'}{\mathcal{H}_0} \tag{17.26}$$

obeys the differential equation

$$\varphi'' + k^2\varphi - \frac{Z''}{Z}\varphi = 0 \tag{17.27}$$

So φ is clearly gauge invariant since it is related to the gauge invariant potential Θ!

Notice $\varphi \propto a\delta\phi$
Naively: $a\delta\phi$ would have been our Gaussian field theory variable to quantise; indeed, ignoring mass terms

$$S = -\frac{1}{2}\int \mathrm{d}^4x\sqrt{g}(\nabla\phi)^2 \tag{17.28}$$

with $\phi = \phi_0 + \delta\phi$.

$$(\nabla\phi)^2 = (\nabla\phi_0)^2 + 2\nabla\phi_0\,\nabla\delta\phi + (\nabla\delta\phi)^2 \tag{17.29}$$

The terms $\propto \phi_0$ drop out by the background equations of motion, leaving

$$S = -\frac{1}{2}\int \mathrm{d}^4x\sqrt{g}(\nabla\delta\phi)^2 \tag{17.30}$$

So: with $\varphi = a\delta\phi$ we would rewrite the theory as

$$S = \frac{1}{2}\int \mathrm{d}\eta\,\mathrm{d}^3\vec{x}\left(\varphi'^2 - (\vec{\nabla}\varphi)^2 + \frac{a_0''}{a_0}\varphi^2\right) \tag{17.31}$$

Note that

$$\frac{Z''}{Z} = \frac{\left(\frac{a_0\phi_0'}{\mathcal{H}_0}\right)''}{\frac{a_0\phi_0'}{\mathcal{H}_0}} = \frac{a_0''}{a_0} + \underbrace{f(\epsilon,\eta)}_{\text{slow roll parameters!}} \tag{17.32}$$

So, in fact we find

$$S = \frac{1}{2}\int \mathrm{d}\eta\mathrm{d}^3\vec{x}\left(\varphi'^2 - (\vec{\nabla}\varphi)^2 + \frac{Z''}{Z}\varphi^2\right) \tag{17.33}$$

because of the *back-reaction.*

$a_0\delta\phi$: not gauge invariant!
But gauge variations are fixed by the term $-\frac{a_0\phi_0'}{\mathcal{H}_0}\Phi$!
So now we can, and know how to, quantise φ!

But what is the measure of the density fluctuations induced by the fluctuations of φ? Consider the longitudinal gauge metric

$$\mathrm{d}s^2 = a_0^2\left\{-(1-2\Phi)\mathrm{d}\eta^2 + (1+2\Phi)\mathrm{d}\vec{x}^2\right\} \tag{17.34}$$

The 3D geometry at some η =const is

$$g_3 = a_0^2(1+2\Phi)\mathrm{d}\vec{x}^2 \tag{17.35}$$

So the 3D metric has been perturbed by Φ which arose in response to $\delta\phi$; we can indeed show that

$$\Phi = \frac{1}{2M_p^2}\frac{{\phi_0'}^2}{\mathcal{H}_0 k^2}\left(\frac{\varphi}{Z}\right)' \tag{17.36}$$

The perturbation of the 3D curvature can be computed straightfowardly; the intrinsic curvature at some η =const of the unperturbed geometry is zero. The perturbation induces spatial dependence, since

$$\Phi \propto e^{i\vec{k}\vec{x}} \tag{17.37}$$

Writing

$$\mathrm{d}\hat{s}_3^2 = \Omega^2\mathrm{d}\vec{x}^2 \tag{17.38}$$

where

$$\Omega^2 = a_0^2(1+2\Phi) \tag{17.39}$$

we have (see Wald, appendix D)

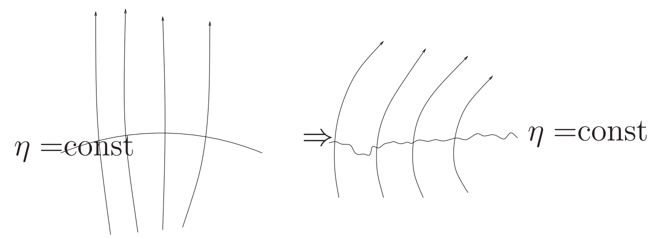
$$R_3 = \mathcal{R} = -\frac{1}{4\Omega^2}\left(4\vec{\nabla}^2\ln\Omega + 2(\vec{\nabla}\ln\Omega)^2\right) \tag{17.40}$$

and so to linear order in Φ

$$\mathcal{R} = -\frac{4}{a_0^2}\vec{\nabla}^2\Phi = \frac{4k^2}{a_0^2}\Phi \tag{17.41}$$

However: in the longitudinal gauge, once we have perturbed the curvature we have ALSO perturbed the density!

So we have BOTH δR_3 and $\delta\rho$, the total scalar perturbation is a combination of these two!



In the longitudinal gauge it is easy to compute field equations, but hard to identify their physical effects!

Instead: change the gauge to the isodensity gauge, where the η =const slices are deformed such that ρ remains unperturbed!

$$\bar{\rho}(\bar{\eta}, x) = \rho(\eta + \delta\eta) + \delta\bar{\rho} \tag{17.42}$$
$$= \rho + \rho'\delta\eta + \delta\bar{\rho} \equiv \rho + \delta\rho \tag{17.43}$$

Requiring that $\delta\bar{\rho} \equiv 0$ yields $\rho'\delta\eta = \delta\rho$ or, therefore

$$\delta\eta = \frac{\delta\rho}{\rho'} \tag{17.44}$$

Now, in the slow roll regime,

$$\rho = V \tag{17.45}$$
$$\therefore \delta\rho = \partial_\phi V\, \delta\phi, \qquad \rho' = \partial_\phi V\, \phi_0' \tag{17.46}$$
$$\therefore \delta\eta = \frac{\delta\phi}{\phi'} \tag{17.47}$$

and so

$$\bar{\eta} = \eta = \frac{\delta\phi}{\phi_0'} \tag{17.48}$$

Thus

$$g = \frac{\delta\phi}{\phi_0'} \tag{17.49}$$

Thus, in this gauge we find

$$\bar{\Phi} = \Phi - \mathcal{H}\frac{\delta\phi}{\phi_0'} \tag{17.50}$$

$$\bar{\Psi} = -\Phi - \left(\frac{\delta\phi}{\phi_0'}\right)' - \mathcal{H}_0\frac{\delta\phi}{\phi_0'} \tag{17.51}$$

$$\bar{B} = \frac{\delta\phi}{\phi_0'} \tag{17.52}$$

so

$$\begin{aligned} ds^2 = a_0^2\Big\{ - \left(1 - 2\Phi - 2\left(\frac{\delta\phi}{\phi_0'}\right)' - 2\mathcal{H}_0\frac{\delta\phi}{\phi_0'}\right) d\eta^2 + 2\frac{\nabla_k\delta\phi}{\phi_0'}dx^k\, d\eta \\ + \left(1 + 2\Phi - 2\mathcal{H}_0\frac{\delta\phi}{\phi_0'}\right) d\vec{x}^2\Big\} \end{aligned} \tag{17.53}$$

Note: $\bar{\Phi} = \bar{\Theta}$!

In this gauge, the curvature perturbation of the η =const surfaces is now

$$\mathcal{R} = \frac{4k^2}{a_0^2}\hat{\Theta} \tag{17.54}$$

The density contrast is $\delta\rho/\rho = \mathcal{R}/R_4$, and using $R_4 = 12H^2$,

$$\frac{\delta R}{R} = \frac{k^2}{3H^2a_0^2}\hat{\Theta} \tag{17.55}$$

or, in terms of the canonically normalised scalar field $\hat{\Theta} = -\varphi/Z$

$$\frac{\delta R}{R} = -\frac{k^2}{3H^2a_0^2}\frac{\varphi}{Z} \tag{17.56}$$

The conventional lore expresses the density contrast in terms of the power spectrum

$$P(\vec{k})\delta^{(3)}(\vec{k} - \vec{q}) = \frac{k^3}{2\pi^2}\langle\Theta_{\vec{k}}\Theta^\dagger_{\vec{q}}\rangle \tag{17.57}$$

so then, since $z = \frac{a_0\phi_0'}{\mathcal{H}_0} = \frac{a_0\dot{\phi}_0}{H_0}$

$$P(\vec{k})\delta^{(3)}(\vec{k} - \vec{q}) = \frac{k^3}{2\pi^2}\left(\frac{H_0}{\dot{\phi}_0}\right)^2\left\langle\frac{\varphi_{\vec{k}}}{a}\frac{\varphi^\dagger_{\vec{q}}}{a}\right\rangle \tag{17.58}$$

Thus

$$P(k) = \left(\frac{H_0}{\dot{\phi}_0}\right)^2 ||\frac{\varphi_k}{a}||^2 \tag{17.59}$$

or

$$P(k) = \left(\frac{H_0}{\dot{\phi}_0}\right)^2 \left(\frac{H_0}{2\pi}\right)^2 \tag{17.60}$$

So the scalar density perturbations are defined by

$$\frac{\delta\rho}{\rho} = \frac{1}{2\pi}\frac{H_0^2}{\dot{\phi}_0} \tag{17.61}$$

We are now ready to apply this to various models of inflation and subject them to observational tests.

Chapter 18

Lecture 18

18.1 Towards models of inflation

Thus: during inflation, to leading order, the cosmological dynamics can be approximated by the inflaton zero mode dynamics in a spatially flat FRW background:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2 \mathrm{d}\vec{x}^2 \tag{18.1}$$

$$3H^2 = \frac{1}{M_p^2}\left(\frac{\dot{\phi}^2}{2} + V\right) \tag{18.2}$$

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V}{\partial \phi} = 0 \tag{18.3}$$

The leading contribution of the inhomogeneous modes in the production of the inhomogeneous curvature perturbations, characterised by the inhomogeneous imprint in the background geometry, given by

$$\Theta = \Phi - \frac{H_0}{\dot{\phi}_0}\delta\phi \propto \mathrm{const} \times e^{i\vec{k}\vec{x}} \tag{18.4}$$

and the surviving r.m.s. curvature perturbation amplitude

$$\frac{\delta\rho}{\rho} = \frac{1}{2\pi}\frac{H_0}{\dot{\phi}_0}\delta\phi = \frac{1}{2\pi}\frac{H_0^2}{\dot{\phi}_0} \tag{18.5}$$

In general the inflationary regime occurs when the dynamics obey slow roll conditions

$$\epsilon = \frac{3\dot{\phi}^2}{2V} \ll 1 \tag{18.6}$$

$$\eta = -\frac{\ddot{\phi}}{H\dot{\phi}} \ll 1 \tag{18.7}$$

We can recast these conditions as the requirements which the inflaton potential $V(\phi)$ must satisfy. Indeed, consider the field equations and assume that ϵ, $\eta \ll 1$; find the slow roll solutions and then check their consistency with the slow roll requirements iteratively.

In the slow roll regime, $\eta, \epsilon \ll 1$:

$$3H^2 = \frac{V}{M_p^2} \tag{18.8}$$

$$3H\dot{\phi} + \frac{\partial V}{\partial \phi} = 0 \tag{18.9}$$

Thus

$$H = \frac{1}{M_p}\sqrt{\frac{V}{3}} \tag{18.10}$$

and from $\dot{\phi} = -\partial_\phi V/3H$:

$$\dot{\phi} = -\frac{M_p}{\sqrt{3V}}\frac{\partial V}{\partial \phi} \tag{18.11}$$

Thus

$$\epsilon = \frac{3\dot{\phi}^2}{2V} = \frac{3}{2V}\frac{M_p^2}{3V}\left(\frac{\partial V}{\partial \phi}\right)^2 = \frac{M_p^2}{2}\frac{\left(\frac{\partial V}{\partial \phi}\right)^2}{V^2} \tag{18.12}$$

$$\eta = -\frac{\ddot{\phi}}{H\dot{\phi}} = \frac{\ddot{\phi}}{\frac{1}{M_p}\sqrt{\frac{V}{3}}\frac{M_p}{\sqrt{3V}}\frac{\partial V}{\partial \phi}} = \frac{3\ddot{\phi}}{\frac{\partial V}{\partial \phi}} \tag{18.13}$$

But we can rewrite $\ddot{\phi}$ as follows

$$\ddot{\phi} = -\frac{M_p}{\sqrt{3V}}\frac{\partial^2 V}{\partial \phi^2}\dot{\phi} + \frac{M_p}{2\sqrt{3}V^{3/2}}\left(\frac{\partial V}{\partial \phi}\right)^2\dot{\phi} \tag{18.14}$$

$$= \frac{M_p^2}{2V}\frac{\partial V}{\partial \phi}\frac{\partial^2 V}{\partial \phi^2} - \frac{M_p}{6V^2}\left(\frac{\partial V}{\partial \phi}\right)^3 \tag{18.15}$$

$$= \frac{M_p^2}{2V}\frac{\partial V}{\partial \phi}\left(\frac{\partial^2 V}{\partial \phi^2} - \frac{M_p}{6V^2}\left(\frac{\partial V}{\partial \phi}\right)^2\right) \tag{18.16}$$

Therefore

$$\eta = \frac{M_p^2}{V}\left(\frac{\partial^2 V}{\partial \phi^2} - \frac{1}{2V}\left(\frac{\partial V}{\partial \phi}\right)^2\right) \tag{18.17}$$

Finally

$$\epsilon = \frac{M_p^2}{2}\frac{\left(\frac{\partial V}{\partial \phi}\right)^2}{V^2} \ll 1 \tag{18.18}$$

$$\eta = \frac{M_p^2}{V}\frac{\partial^2 V}{\partial \phi^2} - \epsilon \ll 1 \tag{18.19}$$

Thus, the conditions an inflationary potential must satisfy to admit a slow roll regime are

$$\left|\frac{\partial V}{\partial \phi}\right| \ll \frac{\sqrt{2}}{M_p}V \tag{18.20}$$

and

$$\frac{\partial^2 V}{\partial \phi^2} \ll \frac{V}{M_p^2} + \frac{1}{2V}\left(\frac{\partial V}{\partial \phi}\right)^2 \tag{18.21}$$

i.e.

$$\left|\frac{\partial V}{\partial \phi}\right| \ll \frac{\sqrt{2}}{M_p}V \tag{18.22}$$

$$\frac{\partial^2 V}{\partial \phi^2} \ll \frac{V}{M_p^2} \tag{18.23}$$

For many potentials the condition (18.23) is more constraining.

Recall that since

$$V(\phi) = \frac{m^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4 + \dots \tag{18.24}$$

where we have cancelled by hand $V_0 = V(0)$ since we want to prevent a large cosmological constant in the minimum of the potential, which we conveniently choose to be at $\phi = 0$, we see that

$$\left.\frac{\partial^2 V}{\partial \phi^2}\right|_{\phi=0} = m^2 \tag{18.25}$$

During inflation $V/M_p^2 = 3H^2$ and so we find that we must have

$$m^2 \ll H^2 \tag{18.26}$$

to be in the slow roll regime. In other words, in order to admit slow roll inflation the potential MUST be sufficiently flat!

Inflation terminates when either of the slow roll parameters become of order unity; then either the field acceleration $\ddot{\phi}$ or the kinetic energy $\dot{\phi}^2$ become too large to ignore.

18.1.1 Density fluctuations

The density fluctuations which are produced are

$$\frac{\delta\rho}{\rho} = \frac{1}{2\pi}\frac{H^2}{\dot{\phi}}. \tag{18.27}$$

Using the slow roll conditions and resorting to the slow roll equations to rewrite those we get

$$\frac{\delta\rho}{\rho} = \frac{1}{2\pi}\frac{\frac{V}{3M_p^2}}{\frac{M_p}{\sqrt{3V}}\frac{\partial V}{\partial \phi}} = \frac{1}{2\pi}\frac{V^{3/2}}{\sqrt{3}M_p^3\frac{\partial V}{\partial \phi}} \tag{18.28}$$

Thus

$$\frac{\delta\rho}{\rho} = \frac{1}{\sqrt{12}\pi}\frac{V^{3/2}}{M_p^3\frac{\partial V}{\partial \phi}} \tag{18.29}$$

at ~ 50 e-folds before the end of inflation this must be about 2×10^{-5} in order to match the COBE normalisation conditions.

$$\left.\frac{\delta\rho}{\rho}\right|_{50} \sim 2 \times 10^{-5} \tag{18.30}$$

observational input!

Note what happens with scales:

$$V = 3H^2 M_p^2 \tag{18.31}$$
$$\Rightarrow V^{3/2} = \sqrt{27} H^3 M_p^3 \tag{18.32}$$

and

$$\frac{\delta\rho}{\rho} = \sqrt{\frac{27}{12\pi^2}} \frac{H^3}{\frac{\partial V}{\partial\phi}} \tag{18.33}$$

A natural normalisation scale is M_p:

$$\frac{\delta\rho}{\rho} = \sqrt{\frac{27}{12\pi^2}} \frac{(H/M_p)^3}{\left(\frac{\frac{\partial V}{\partial\phi}}{M_p^3}\right)} \tag{18.34}$$

Although $(H/M_p)^3 \ll 1$ (actually $\lesssim 10^{-17}$), during inflation we have $\frac{\partial V}{\partial\phi}/M_p^3 \ll 1$ also, lifting the ratio $\delta\rho/\rho$ towards 10^{-5}!

Indeed

$$\frac{\partial V}{\partial\phi} = \sqrt{2\epsilon}\frac{V}{M_p} = \sqrt{2\epsilon} 3H^2 M_p \tag{18.35}$$
$$\therefore \frac{\delta\rho}{\rho} = \frac{1}{2\sqrt{2}\pi} \frac{H}{\sqrt{\epsilon} M_p} \tag{18.36}$$

so $H/M_p \lesssim 10^{-6}$, but this gets amplified by $\sqrt{\epsilon} < 1$!

Now if $V^{3/2}/\frac{\partial V}{\partial\phi}$ were *exactly* constant, $\delta\rho/\rho$ would have been independent of time, and therefore of the momentum k of the perturbation.

Recall: the amplitude of the perturbation is set by its value at the horizon exit, which is set by

$$k^2 = \frac{1}{\eta^2} = H^2 a^2 \tag{18.37}$$

so when we solve for ϕ, H as functions of t we find $\frac{\delta\rho}{\rho}(t)$. We can eliminate t for k using the horizon exit condition $k = aH$. Generally during exponential inflation this gives

$$k = H a_0 e^{Ht} \tag{18.38}$$

and so

$$t = \frac{1}{H} \ln \frac{k}{k_\star} \tag{18.39}$$

Thus

$$\frac{\delta\rho}{\rho}(k) = \left(\frac{\delta\rho}{\rho}(k_\star)\right) \ln \frac{k}{k_\star} \tag{18.40}$$

Since $\delta\rho/\rho$ is nearly constant the dependence on $\ln \frac{k}{k_\star}$ is VERY weak and so we expect it to obey

$$\frac{\delta\rho}{\rho} = \left(\frac{\delta\rho}{\rho}\right)_0 (\ln k)^{\frac{n_s-1}{2}} \tag{18.41}$$

where n_s is the *spectral index*, given by

$$n_s = 1 + 2\frac{\mathrm{d}\ln\frac{\delta\rho}{\rho}}{\mathrm{d}\ln k} \tag{18.42}$$

Observationally

$$n_s = 1 \pm 0.1 \qquad \text{(conservative)} \tag{18.43}$$

Very nearly scale invariant – inflation was reasonably closely approximated by de Sitter.

During inflation, the tensor modes $\propto h_{kl}$ are also produced – these are the primordial gravity waves. They contribute to $\delta\rho/\rho$ also, although their amplitude is smaller by a power of $\sqrt{\epsilon}$ due to the details of dynamics.

With $h_{kl} \sim h\epsilon_{kl}$ we have

$$\left.\frac{\delta\rho}{\rho}\right|_T = \frac{\langle h\rangle}{M_p} = \frac{\sqrt{\langle 0|h_k h_q^\dagger|0\rangle}}{M_p} \tag{18.44}$$

$$\approx \frac{H}{M_p} \approx \sqrt{\epsilon}\left.\frac{\delta\rho}{\rho}\right|_S \tag{18.45}$$

i.e. we have

$$\epsilon = \frac{\left(\left.\frac{\delta\rho}{\rho}\right|_T\right)^2}{\left(\left.\frac{\delta\rho}{\rho}\right|_S\right)^2}, \tag{18.46}$$

up to integration constants.

On the other hand, the spectral index n_s is

$$\frac{n_s - 1}{2} = \frac{\mathrm{d}\ln\frac{\delta\rho}{\rho}}{\mathrm{d}\ln k} \tag{18.47}$$

From $t = 1/H \ln\frac{k}{k_\star}$ and $\frac{\delta\rho}{\rho} \sim \frac{H^2}{\dot{\phi}}$

$$\frac{n_s - 1}{2} \approx \frac{\mathrm{d}\ln\frac{H^2}{\dot{\phi}}}{\mathrm{d}(Ht)} \sim \frac{1}{H^2}\frac{1}{H}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{H^2}{\dot{\phi}}\right) \tag{18.48}$$

$$= \frac{\dot{\phi}}{H^3}\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{3\dot{\phi}}\frac{V}{M_p^2}\right] \approx \frac{\frac{\partial V}{\partial\phi}\dot{\phi}}{3M_p^2 H^3} \tag{18.49}$$

$$= \frac{\frac{\partial V}{\partial\phi}}{V}\frac{\frac{M_p}{\sqrt{3V}}\frac{\partial V}{\partial\phi}}{\frac{1}{M_p}\sqrt{\frac{V}{3}}} = M_p^2\frac{\left(\frac{\partial V}{\partial\phi}\right)^2}{V^2} = 2\epsilon \tag{18.50}$$

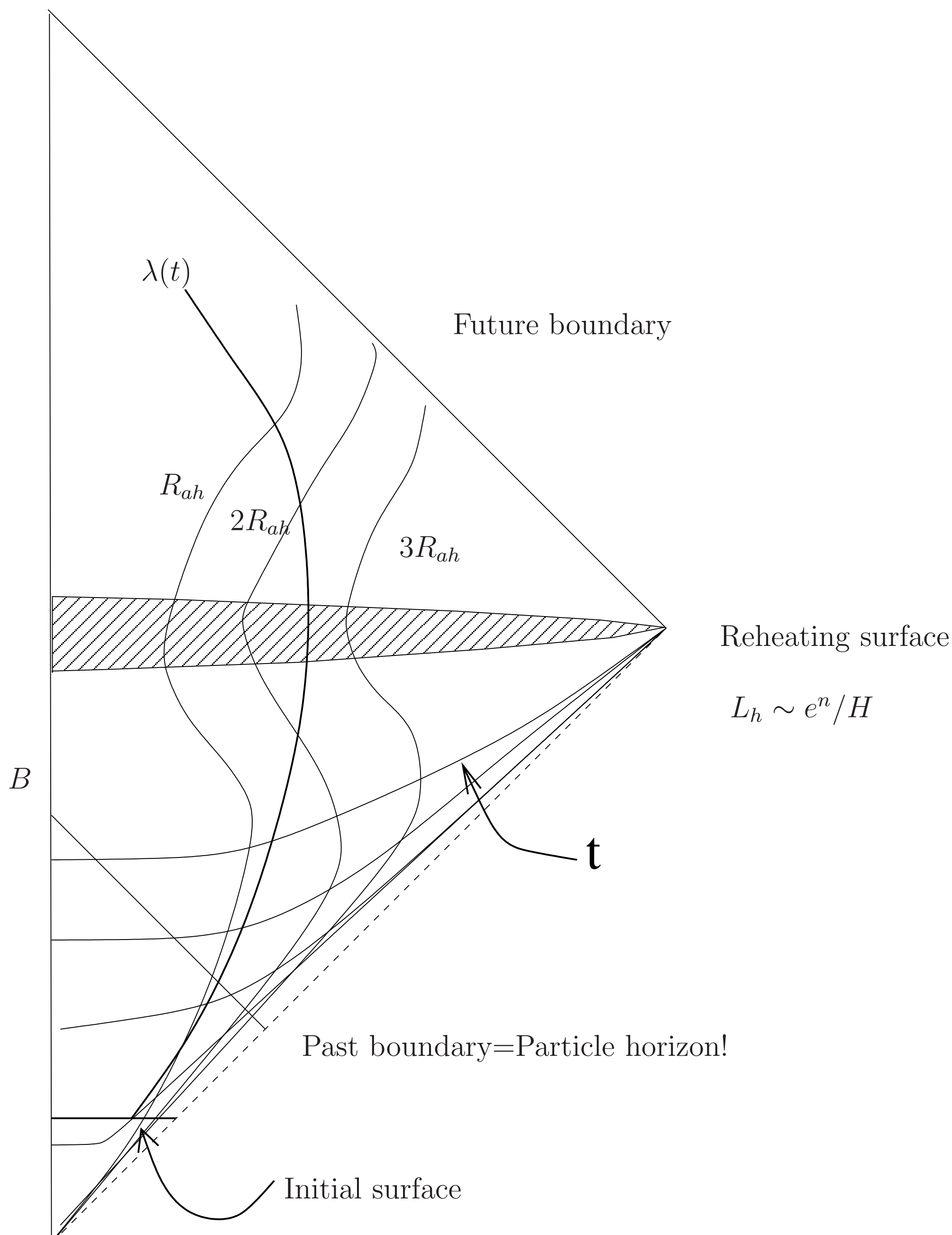
Thus:

$$n_s = 1 + c\frac{\left(\left.\frac{\delta\rho}{\rho}\right|_T\right)^2}{\left(\left.\frac{\delta\rho}{\rho}\right|_S\right)^2} \tag{18.51}$$

Inflationary consistency condition! Arises because in the slow roll regime we have fewer integration parameters and so observables are mutually related → a useful check of inflation onces tensors are observed!

Chapter 19

Lecture 19



$$ds^2 = -dt^2 + a^2 d\vec{x}^2 \tag{19.1}$$

B = past-oriented lightcone: $R_B \sim \frac{1}{H_0}$!

Chapter 20

Lecture 20

20.1 Desperately seeking the inflaton

Who is the inflaton?

To date many models have been developed. We will consider a number of them. In chronological order, they are

1. *Old inflation*
2. *New inflation*
3. *Chaotic inflation*
4. *Modular inflation*
5. *Power law inflation*
6. Hybrid inflation
7. Extended inflation
8. Super-extended inflation
9. $R + R^2$ *inflation*

We will cover a number of models, focussing on

- Inflationary dynamics
- Generation of perturbations
- ending inflation and reheating

20.2 Old inflation

(Reference for this section: Guth [17])

Idea: consider a scalar field with

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - V(\phi) \tag{20.1}$$

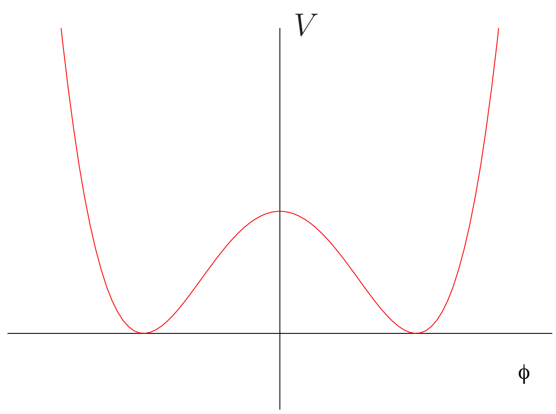
where $V(\phi)$ is some potential with spontanous symmetry breaking:

$$V(\phi) = \frac{\Lambda}{4}\phi^4 - \frac{1}{2}\mu^2\phi^2 + V_0 \tag{20.2}$$

Fine-tune V_0 such that the potential at the minimum is zero. Easiest to do by completing the square:

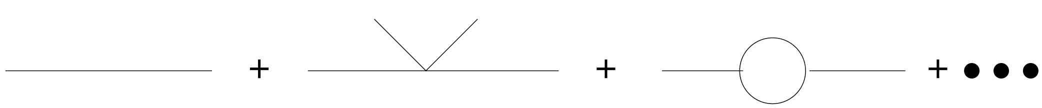
$$V(\phi) = \frac{\Lambda}{4}\left(\phi^2 - \frac{\mu^2}{\Lambda}\right)^2 \tag{20.3}$$

which show the minima are at $\phi = \pm\mu/\sqrt{\Lambda}$ and $V_0 = \mu^4/4\Lambda$.



Now: assume that the universe is at some temperature T, in equilibrium and that ϕ couples to the thermal bath of particles.

Then there will arise temperature corrections to the potential, coming from the interactions with the background:



The 1-loop connected potential is

$$V_T(\phi) = V(\phi) + \frac{T}{2\pi^2}\int_0^\infty \mathrm{d}k\, k^2 \ln\left[1 - \exp\left(-\frac{k^2 + \frac{\partial^2 V}{\partial\phi^2}}{T}\right)\right] \tag{20.4}$$

Evaluating $m_\phi^2 = \partial^2 V/\partial\phi^2$ at the minimum and assuming that $T \gg m_\phi^2$, by expanding the integrand in powers of $(m_\phi/T)^2$ we find

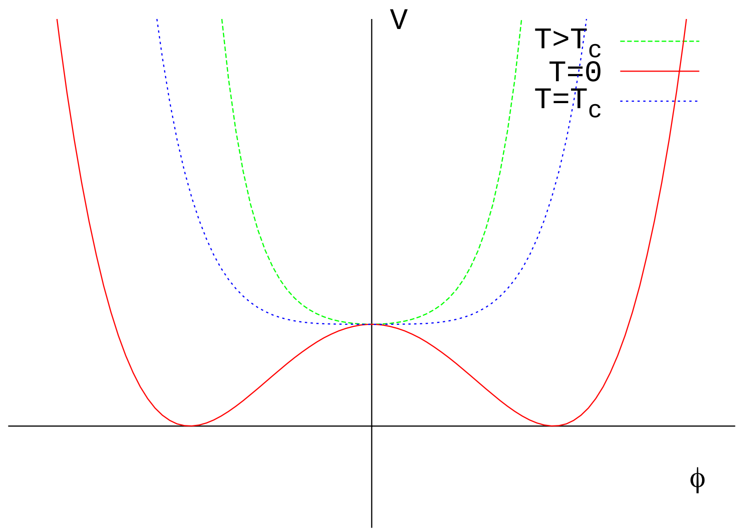
$$V_T(\phi) = V(\phi) + \frac{1}{24}\frac{\partial^2 V}{\partial\phi^2}T^2 + \frac{\Lambda}{8}T^2\phi^2 - \frac{1}{90}\pi^2 T^4 \tag{20.5}$$

$$= \frac{\Lambda}{4}\phi^4 + \frac{1}{2}\left(\frac{\Lambda}{4}T^2 - \mu^2\right)\phi^2 + V_0 - \frac{1}{24}\mu^2 T^2 - \frac{1}{90}\pi^2 T^4 + V_0 \tag{20.6}$$

So the effective mass at non-zero temperature is, around $\phi = 0$ is

$$m_{\phi_{eff}}^2 = \frac{\Lambda}{4}T^2 - \mu^2 \tag{20.7}$$

Temperature effects change the structure of the potential!



Finding the critical temperature:

$$T_c: \qquad m_{\phi_{eff}} = 0 \Rightarrow T_c = \frac{2\mu}{\sqrt{\Lambda}} \tag{20.8}$$

In the case of theories with spontaneously broken local gauge symmetries the situation may be more complicated because there may have been additional phase transitions [8].

Idea for inflation: at $T \gg m_\phi$, the symmetry is restored. By dynamics, assuming $m_\phi \gg H$, (which is consistent while $T < M_p$)

$$T \gg m_\phi \gg H \sim \frac{T^2}{M_p} \tag{20.9}$$

the field ϕ is trapped at $\phi = 0$. This is the minimum of the potential at high T.

Then the universe cools. When

$$T^4 \sim V_0 \tag{20.10}$$

or, in other words,

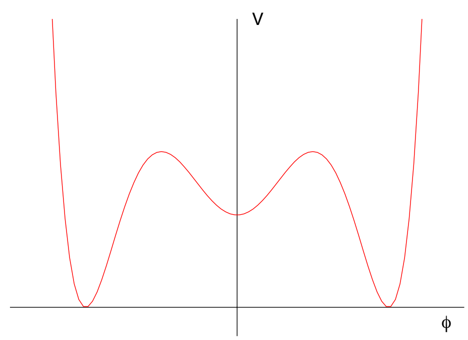
$$T \sim \left(\frac{\mu^4}{\Lambda}\right)^{1/4} \tag{20.11}$$

the potential $V(\phi = 0) = V_0$ starts to dominate. Inflation begins, while the field is still trapped at $\phi = 0$.

More general than ϕ^4

The universe eventually expands unimpeded until T drops below the critical temperature T_c where the symmetry is broken.

Further cooling by expansion brings the universe into the "supercooled" state. The field now sits in the false vacuum:



The field can now tunnel through the barrier from the false minimum at $\phi = 0$ to the true minimum ϕ_{min}[1]

In regions of space where this happens inflation ceases and the vacuum energy decays into particles, releasing energy.

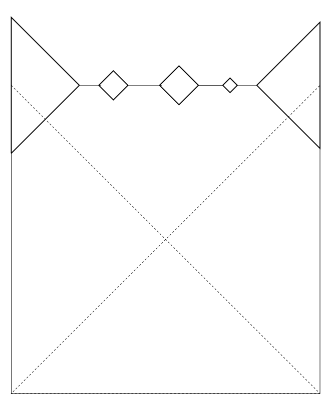
However, the problem with old inflation is that it never ends properly! (The real problem is not that it does not end, but that the size of the "bubbles" it creates are not big enough for the observed universe).

The regions of space where the tunnelling completed look like bubbles of Minkowski space enveloped in the de Sitter space environment. In the de Sitter space environment the field is still stuck in the "false minimum".

The bubbles expand at the speed of light – but the environment is STILL growing exponentially quickly!

So the interspatial distance between two typical adjacent bubbles grows – they never meet and connect!

So the regions at the Minkowski space inside the bubbles never grow big enough to be the size of our current universe!



In the old inflation scenario inflation never completes properly and so the resulting universe is a big de Sitter space, rapidly inflating which happens to

[1]Damien: A nice pair of papers on the rate of vacuum decay are from Coleman [7].

be sparsely populated by small islands of FRW universes, separated by large domain walls.

$\Rightarrow$ Very inhomogenous final state[2]

GRACEFUL EXIT PROBLEM!

In order to have a chance for succeeding, inflation must terminate gracefully, producing only desirable relics instead of domain walls and other beasts.

This was an interesting, educational failure.

[2]Damien: I think this is a problem only because the final state is inhomogenous and too small. I do not think we care if the universe is inhomogenous, provided that it occurs outside our causal patch.

Chapter 21

Lecture 21

21.1 New inflation

(References for this section are Albrecht & Steinhardt [18] and Linde [19])

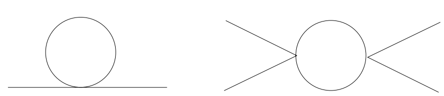
Consider again the effective potential for a scalar field ϕ:

$$V(\phi) = \frac{\Lambda}{4}\left(\phi^2 - \frac{\mu^2}{\Lambda}\right)^2 + V_{1\text{ loop}} \tag{21.1}$$

where

$$V_{1\text{ loop}} = \frac{A}{2}\phi^2 + \frac{B}{4}\phi^4 + \frac{1}{2}\int \frac{\mathrm{d}^4 k}{(2\pi)^4} \ln\sqrt{1+\frac{m^2}{k^2}} \tag{21.2}$$

where A, B are mass and coupling counterterms regulating divergences in



etc

and $\int$ = sum of zero-point energies.

One can determine A and B by imposing the renormalisation conditions

$$\left.\frac{\partial^2 V}{\partial \phi^2}\right|_{\phi=0} = 0, \qquad \left.\frac{\partial^4 V}{\partial \phi^4}\right|_{\phi=M} = 6\Lambda \tag{21.3}$$

corresponding to a choice of subtraction scheme.

This then fixes the potential to be

$$V(\phi) = \frac{\Lambda}{4}\phi^4 + \frac{m^4}{64\pi^2}\left(\ln\frac{m^2}{M^2} - \frac{25}{6}\right) + \frac{\mu^4}{4\Lambda}. \tag{21.4}$$

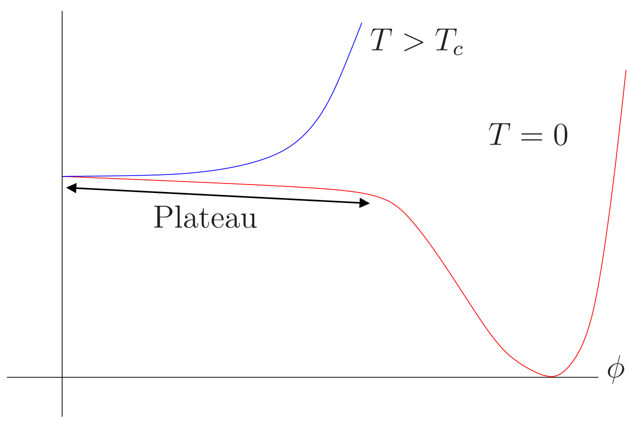
Here m is the bare mass of a particle in the loop and M is th erenormalisation scale.

When $m^2 = 3\lambda\eta^2$, this is precisely the Coleman-Weinberg potential with radiatively induced symmetry breaking.

So: consider a potential of the form

$$V(\phi) = \frac{\Lambda}{4}\phi^4 + \frac{g}{4}\phi^4\left(\ln\frac{\phi^2}{M^2} - f\right) + V_0 \tag{21.5}$$

such that $V(\phi_{\text{min}}) = 0$. Below we show the potential with temperature corrections included:



Example of slow-roll inflation!

1. At high temperatures, the field has mass $m \sim T \gg T^2/M_p = H$ so it is quickly trapped in the minimum $\phi = 0$.

2. As the universe cools, $T^4 < V_0$ and inflation begins

3. The field wants to do to the minimum but it has ways to go since it finds itself on top of a *very* flat plateau.

The width of the plateau controls the duration of inflation. We have

$$\frac{\partial^2 V}{\partial \phi^2} = m(\phi)^2 \sim \text{constant} < H^2 \tag{21.6}$$

so in the slow roll regime:

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\partial^2 V}{\partial \phi^2}\phi = 0 \tag{21.7}$$

reduces to

$$\dot{\phi} \approx -\frac{m^2}{3H}\phi \tag{21.8}$$

and so $\phi = \phi_0 \exp(\frac{m^2}{3H}\Delta t)$. Note these approximations break down after about 1 or 2 e-folds, i.e. after a time

$$\Delta t \sim \frac{3H}{m^2} \tag{21.9}$$

During this regime,

$$a = e^{H\,\Delta t} \tag{21.10}$$

so since $H\,\Delta t \geq 70$, and field moves away from the slow roll regime. Given the amount of time we can trust this solution, we require that

$$\frac{3H^2}{m^2} \gtrsim 70 \tag{21.11}$$

This is the cost we must pay to have sufficient inflation; $m^2 = \frac{\partial^2 V}{\partial \phi^2}$

$$\frac{\partial^2 V}{\partial \phi^2} \lesssim \frac{3}{70}H^2 \tag{21.12}$$

Measure of the flatness of the potential!

We can now calculate the density perturbation:

$$\frac{\delta\rho}{\rho} = \frac{H}{\dot{\phi}}\delta\phi = \frac{H^2}{2\pi\dot{\phi}} \tag{21.13}$$

In the slow roll regime,

$$H^2 = \frac{V}{3M_p^2} \tag{21.14}$$

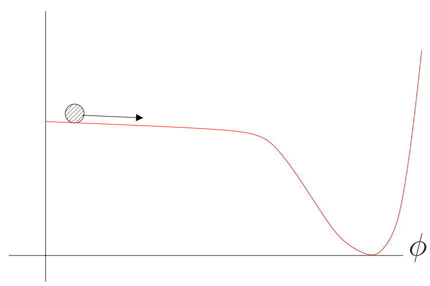
$$\dot{\phi} = -\frac{1}{3H}\frac{\partial V}{\partial \phi} \tag{21.15}$$

So

$$\frac{\delta\rho}{\rho} = \frac{1}{2\pi}\frac{3H^3}{\frac{\partial V}{\partial \phi}} = \frac{1}{2\sqrt{3}\pi}\frac{V^{3/2}}{M_p^3\frac{\partial V}{\partial \phi}} \tag{21.16}$$

The potential is approximately

$$V = V_0 + \frac{m^2}{2}\phi^2 - \frac{\Lambda}{4}\phi^4 \tag{21.17}$$



Now from

$$3H\dot{\phi} = \Lambda\phi^3 \tag{21.18}$$

and $H \sim$ constant,

$$\frac{\dot{\phi}}{\phi^3} \approx +\frac{\Lambda}{3H} = -\frac{1}{2}\left(\frac{1}{\phi^2}\right) \tag{21.19}$$

so

$$\frac{1}{\phi^2} - \frac{1}{\phi_0^2} = -\frac{2\Lambda}{3H}t \tag{21.20}$$

and hence, ignoring $1/\phi_0^2$,

$$\phi^2 \approx -\frac{3H}{2\Lambda t} \tag{21.21}$$

Then $V' \approx \Lambda\phi^3$ and so

$$\frac{\delta\rho}{\rho} = \frac{3}{2\pi}\frac{H^3}{\Lambda\phi^3} \sim \frac{3}{2\pi\Lambda}\frac{H^3}{\left(\frac{3H}{2\Lambda t}\right)^{3/2}} \tag{21.22}$$

But: t can be determined from the horizon crossing condition:

$$\lambda = \frac{a}{k} = \frac{1}{H} \Rightarrow e^{Ht} = \frac{k}{H} \tag{21.23}$$

so

$$t \sim \frac{1}{H}\ln\frac{k}{H} \tag{21.24}$$

So:

$$\frac{\delta\rho}{\rho} \sim \frac{1}{\pi}\sqrt{\frac{2\Lambda}{3}}\frac{H^3}{\left(H^2(\ln\frac{k}{H})^{-1}\right)^{3/2}} \tag{21.25}$$

i.e

$$\frac{\delta\rho}{\rho} = \sqrt{\frac{2}{3}}\frac{1}{\pi}\sqrt{\Lambda}\left(\ln\frac{k}{H}\right)^{3/2} \tag{21.26}$$

. So for a typical galaxy

$$\ln\frac{k}{H} \sim 100 \tag{21.27}$$

so

$$\frac{\delta\rho}{\rho} \sim \mathcal{O}\left(100\right)\sqrt{\Lambda} \tag{21.28}$$

For SU(5) grand unified theories $\Lambda \sim 1 \rightarrow \delta\rho/\rho \gg 10^{-5}$!!!

Even if you made an inflation model with $\Lambda \sim 10^{-14}$ so that $\delta\rho/\rho$ came out correctly for the galaxies, at short scales $\delta\rho/\rho \sim 1$ again, and we create too many primordial black holes [**?**]!

More concretely, we can see from that the spectrum is not scale invariant. Recall the definition of the scaling index

$$\frac{\delta\rho}{\rho} \sim \left(\ln\frac{k}{H}\right)^{\frac{n_s-1}{2}}. \tag{21.29}$$

New inflation predicts that $n_s = 2$, while CMB data gives us $n_s = 1 \pm 0.1$.

MORAL: If you have a new inflationary model, check $\delta\rho/\rho$ first. It is much harder to get right than the slow roll conditions, and is the thing that kills most inflationary models.

Chapter 22

Lecture 22

22.1 Chaotic inflation

(References for this section are works by Linde [20])

How it "works"

We have now seen two examples of inflation, and they both failed for different reasons. Chaotic inflation fixes these problems, but does so in a slightly awkward way.

- No quantum potential
  Chaotic inflation does not try and justify the form of $V(\phi)$ from some high energy theory. A (pseudo-)justification for this is that we do not understand the dark sector properly, so who is to say that we cannot have these forms of potentials. Ultimately, it is a bottom up approach.

- We need inflation already. More precisely, for chaotic inflation to work we need something that looks like de-Sitter space in the first place. Chaotic inflation tells us how we can get a lot of inflation out of a mild amount to start with.

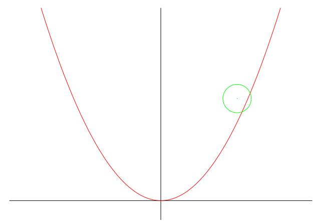
With these caveats in mind, we start to look at chaotic inflation

22.1.1 The model

A very clever observation: consider any polynomial potential

$$V = \frac{\lambda}{n}\phi^n \tag{22.1}$$

For a potential with large λ and small ϕ:



In the limit $\lambda \to 0$, while holding the initial value of V fixed we get

To do this, we need $\phi = (nV/\lambda)^{1/n}$ which gets large as λ decreases.

How does ϕ get this large? Remember the inflationary freezeout. Assume $\partial^2 V/\partial\phi^2 \ll H^2 \to$ relativistic field.

$$\lambda < \frac{1}{H} \qquad \phi = \frac{A}{a}\cos(k\eta + \delta)e^{i\vec{k}\cdot\vec{x}} \tag{22.2}$$

$$\lambda > \frac{1}{H} \qquad \phi = \left(\bar{A} + \frac{\bar{B}}{a^3}\right)e^{i\vec{k}\cdot\vec{x}} \tag{22.3}$$

Inside the Hubble patch: $\lambda > 1/H$ DO NOT fluctuate in time (and spatial variation is negligable compared to the Hubble scale). Namely, once they are pushed out of the horizon, the fluctuations freeze and are practically indistinguishable from the zero mode at short times and scales.

So the zero mode effectively builds up in some region of space!

Indeed, RMS value of $\delta\phi_k$ is

$$\delta\phi_k = \frac{H}{2\pi}! \tag{22.4}$$

Fluctuations can add up!

So the average deviation of the field can be defined by the square root of the 2-point function

$$||\phi_k(\eta)||^2 = \langle\phi(x)\phi(x)\rangle \tag{22.5}$$

$$= \int \frac{\mathrm{d}^3k}{(2\pi)^3}|\phi_k(\eta)|^2 \tag{22.6}$$

where

$$\phi_k(\eta) = \frac{1}{a\sqrt{2k}}\left(1 - \frac{i}{k\eta}\right)e^{ik\eta} \tag{22.7}$$

so

$$||\phi||^2 = \frac{1}{(2\pi)^3}\int\frac{\mathrm{d}^3k}{2k}\left(1+\frac{1}{k^2\eta^2}\right)\frac{1}{a^2} \quad = \frac{1}{(2\pi)^3}\int\frac{\mathrm{d}^3k}{2k}\left(1+\frac{1}{k^2\eta^2}\right)H^2\eta^2 \tag{22.8}$$

Hence

$$||\phi||^2 = \frac{H^2}{(2\pi)^3}\int\frac{\mathrm{d}^3k}{2k}\left(\eta^2 + \frac{1}{k^2}\right) \tag{22.9}$$

(Slightly different normalisation than in Linde's book)

Interpreting these terms is a lot easier if we look at the integral in physical momenta space. We really want to do the integral at with limits at fixed *physical* momenta, because otherwise we are saying that the physical scale that our effective field theory (EFT) breaks down is time dependent. We have

$$p = \frac{k}{a} = -H\eta k \tag{22.10}$$

and hence, for fixed times

$$\frac{\mathrm{d}^3p}{p^3} = \frac{\mathrm{d}^3k}{k^3} \tag{22.11}$$

Let us use (22.10) to eliminate η for p, then change the integration to physical momentum space

$$||\phi||^2 = \frac{H^2}{(2\pi)^3}\int \frac{\mathrm{d}^3k}{2k}\left(\frac{p^2}{H^2k^2}+\frac{1}{k^2}\right) \tag{22.12}$$

$$= \frac{1}{(2\pi)^3}\int \frac{\mathrm{d}^3k}{k^3}\left(\frac{p^2}{2}+\frac{H^2}{2}\right) \tag{22.13}$$

$$= \frac{1}{(2\pi)^3}\int \frac{\mathrm{d}^3p}{p^3}\left(\frac{p^2}{2}+\frac{H^2}{2}\right) \tag{22.14}$$

Now:

- The first term $\propto p^2/2$ is the usual zero-point energy contributions
- Renormalise to zero! Otherwise there would be a divergent contribution!

The first term you would get in Minkowski space – the second term is what is different in inflation!

Looking at the difference from the flat space value, we have

$$||\phi(\eta)||^2 = \frac{H^2}{(2\pi)^3}\int \frac{\mathrm{d}^3p}{2p^3} = \frac{H^2}{(2\pi)^2}\int \frac{\mathrm{d}p}{p} \tag{22.15}$$

Pick the limits of integration determined by the beginning of inflation and the moment of measurement. This gives

$$||\phi(\eta)||^2 = \frac{H^2}{(2\pi)^2}\int_{He^{-Ht}}^{H} \frac{\mathrm{d}p}{p} \tag{22.16}$$

$$\approx \frac{H^2}{(2\pi)^2}\ln e^{Ht} \tag{22.17}$$

$$= \frac{H^3t}{(2\pi)^2} \tag{22.18}$$

So: $||\phi|| = \phi_0(t)$ grows linearly by the average superposition of fluctuations. (Not true for arbitrarily late times, can only by pushed up to $\sim H^4$, as after that the size of the fluctuation is so big that we cannot neglect the effect of the finite energy of the "heat bath" anymore.)

In other words: at some regions of initial space, ϕ may be really really big!

Chaotic inflation initial conditions: assume ϕ can be as big as you like as long as the energy density stored in it does not exceed Planckian value M_p^4. The story goes that even if those spots where subdominant to begin with they will dynamically come to dominate the phase space.

22.1.2 Validity of the EFT

Provided we cutoff when $\rho \lesssim M_p^4$ we expect GR to be a good EFT for gravity.

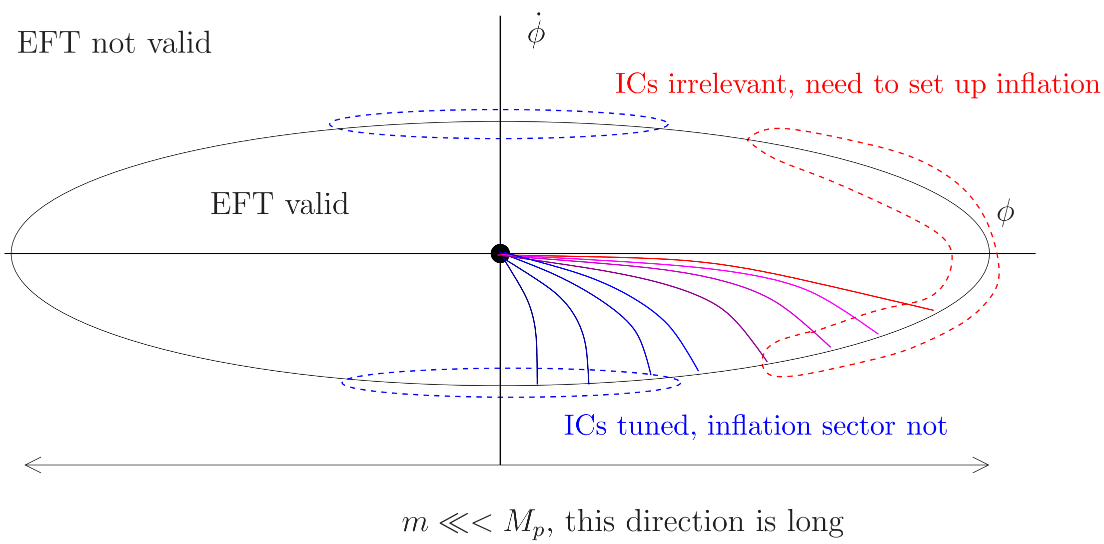
$$\rho = \frac{\dot{\phi}^2}{2} + V(\phi) \tag{22.19}$$

Pick for simplicity $V = m^2\phi^2/2$ and consider the parameter $\Omega \equiv \rho/M_p^4$.

Region where EFT is valid is given by $\Omega \leq 1$, i.e.

$$\frac{\dot{\phi}^2}{2M_p^4} + \frac{m^2\phi^2}{2M_p^4} \leq 1 \tag{22.20}$$

Ellipse!



A note on attractors

A couple of things that we should note about the ellipse. First of all, the ellipse contains all points where we can trust GR (and the evolution of the inflaton) as an EFT. Not all places describe "good evolution", such as any point on the ellipse with $\phi > 0$ and $\dot{\phi} > 0$. This would correspond to a "ball being thrown upward" in mechanical terms, and the evolution of this system would lead us out of the EFT ellipse.

Restricting our attention to the points that remain in the ellipse, we see all of them end with $\phi = \dot{\phi} = 0$. This point is an *stable attractor* in the parlance of dynamical systems. Here (or arbitarily close by) is where we live now.

This is a blessing and a curse. It is a blessing because the final state is a robust prediction of chaotic inflation. It is a curse because we cannot tell what our initial conditions where. We could be a very finely tuned universe that had 1–2 e-folds of inflation (shown in blue), or we could have had an initial state that had the inflation sector finely tuned but is insensitive to the matter initial conditions (shown in red).

Back to inflation . . .

So: to have a long period of inflation, assume that initially

$$\Omega \gg 1 \qquad \text{and} \qquad V \gg \frac{\dot{\phi}^2}{2} \Rightarrow \phi \gg M_p^w \tag{22.21}$$

For $V = m^2\phi^2/2$:

$$3H^2 = \frac{m^2}{2M_p^2}\phi^2 \tag{22.22}$$

$$3H\dot{\phi} = -m^2\phi \tag{22.23}$$

so

$$H = \frac{m}{\sqrt{6}M_p}\phi \tag{22.24}$$

$$\dot{\phi} = -\sqrt{\frac{2}{3}}M_p m \tag{22.25}$$

Thus

$$\phi \approx \phi_0 - \sqrt{\frac{2}{3}} M_p m t \tag{22.26}$$

$$\frac{\dot{a}}{a} = \frac{m}{\sqrt{6} M_p} \left(\phi_0 - \sqrt{\frac{2}{3}} M_p m t \right) \tag{22.27}$$

solving for $a(t)$ yields

$$a \approx a_0 \exp\left(\frac{m}{\sqrt{6} M_p} \left(\phi_0 - \sqrt{\frac{2}{3}} M_p m t \right) \right) \tag{22.28}$$

22.1.3 Slow roll parameters in chaotic inflation

Now $\ddot{\phi} = 0$, so $\eta = -\ddot{\phi}/(H\dot{\phi}) = 0$!. Checking the ϵ slow roll paramter:

$$\epsilon = \frac{3\dot{\phi}^2}{2V} \approx \frac{3}{2\,\frac{m^2\phi^2}{2}} \left(\sqrt{\frac{2}{3}} M_p m \right)^2 \tag{22.29}$$

$$\therefore \epsilon = 2\frac{M_p^2}{\phi^2} \tag{22.30}$$

So as long as $\phi \gg \sqrt{2} M_p$ we are in slow roll!

Initially

$$\frac{m^2\phi^2}{2} \approx M_p^2 \tag{22.31}$$

so

$$\sqrt{2} M_p \le \phi \le \sqrt{2}\frac{M_p}{m} M_p \tag{22.32}$$

is the region where slow roll inflation can occur!

Let us take $\phi_0 \approx \sqrt{2}(M_p/m)M_p$. Then inflation ends when $\phi_0 \approx \sqrt{2} M_p$, thus its duration is determined by $t_\star$ where

$$\sqrt{2} M_p \approx \sqrt{2}\frac{M_p}{m} M_p - \sqrt{\frac{2}{3}} M_p M t_* \tag{22.33}$$

i.e.

$$t_\star = \frac{\sqrt{3}}{m}\left(\frac{M_p}{m} - 1\right) \approx \sqrt{3}\frac{M_p}{m^2} = \sqrt{3}\left(\frac{M_p}{m}\right)^2 t_p^2 \tag{22.34}$$

The total number of e-folds then is

$$\bar{N} = \frac{m}{\sqrt{6}M_p}\left(\phi_0 t_\star - \frac{M_p m}{\sqrt{6}} t_\star^2\right) \tag{22.35}$$

$$\therefore \bar{N} = \frac{M_p^2}{2m^2} + \dots \tag{22.36}$$

If $m \ll M_p$, $N \gg 1$!

Finally let $\mathcal{N}$ be the number of e-folds before the end of inflation

$$\mathcal{N} = \bar{N} - N \tag{22.37}$$

$$= \bar{N} - \ln\left(\frac{a(t)}{a_0}\right) \tag{22.38}$$

Simple algebra then shows that

$$\mathcal{N} = \frac{1}{4}\left(\frac{\phi}{M_p}\right)^2 \tag{22.39}$$

Then density contrast is

$$\frac{\delta\rho}{\rho} = \frac{H^2}{2\pi\dot{\phi}} = \frac{\frac{m^2\phi^2}{6M_p^2}}{2\pi\sqrt{\frac{2}{3}}M_p m} \tag{22.40}$$

$$= \frac{1}{4\sqrt{6}\pi}\frac{m\phi^2}{M_p^3} = \frac{1}{4\sqrt{6}\pi}\frac{m}{M_p}4\mathcal{N} \tag{22.41}$$

so

$$\frac{\delta\rho}{\rho} \approx \frac{\mathcal{N}}{\sqrt{6}\pi}\frac{m}{M_p} \tag{22.42}$$

To get $\delta\rho/\rho \sim 10^{-5}$ about 60 e-folds before the end of inflation for galaxy formation. Using $\mathcal{N}/\sqrt{6}\pi \sim \mathcal{O}(10)$, we need

$$m \sim 10^{-6} M_p \sim 10^{13}\,\text{GeV} \tag{22.43}$$

Then

$$\bar{N} = \frac{1}{2}\left(\frac{M_p}{m}\right)^2 \sim 10^{12}\,\text{GeV!} \tag{22.44}$$

Spectral index

$$n_s = 1 + 2\frac{\mathrm{d}\ln\frac{\delta\rho}{\rho}}{\mathrm{d}\ln k} \tag{22.45}$$

$$\frac{\mathrm{d}\ln\frac{\delta\rho}{\rho}}{\mathrm{d}\ln k} = \frac{\mathrm{d}\ln\mathcal{N}}{\mathrm{d}\ln k} = \frac{k}{\mathcal{N}}\frac{\mathrm{d}\mathcal{N}}{\mathrm{d}k} \tag{22.46}$$

But by the horizon crossing matching conditions

$$\lambda_{\text{wavelength}} = \frac{a}{k} = \frac{1}{H} \to k = Ha = Ha_0 e^{\bar{N}-\mathcal{N}} \tag{22.47}$$

We also have

$$\frac{\mathrm{d}k}{\mathrm{d}\mathcal{N}} = -k \qquad \text{and so: } \frac{\mathrm{d}\mathcal{N}}{\mathrm{d}k} = -\frac{1}{k} \tag{22.48}$$

This implies

$$n_s = 1 - \frac{2}{\mathcal{N}} \tag{22.49}$$

Thus about 60 e-folds before the end of infaltion, $n_s = 0.967$, This is scale invariant (same as WMAP 3 $\pm$ errors!!!!)

Note:

$$\epsilon = 2\frac{M_p^2}{\phi^2} \approx \frac{1}{2\mathcal{N}} \tag{22.50}$$

So at $\mathcal{N} \sim 60$ before the end of inflation (relevant for galaxies) we have

$$\epsilon \ll \frac{1}{120} \ll 1 \tag{22.51}$$

$\Rightarrow$ deep in the slow roll regime.

As inflation ends, ϕ start to slosh around the minimum. Then $m \gg \phi$ since $\phi \ll M_p$ and so the only influence of the cosmological term H is to account for the redshift due to expansion. We can that

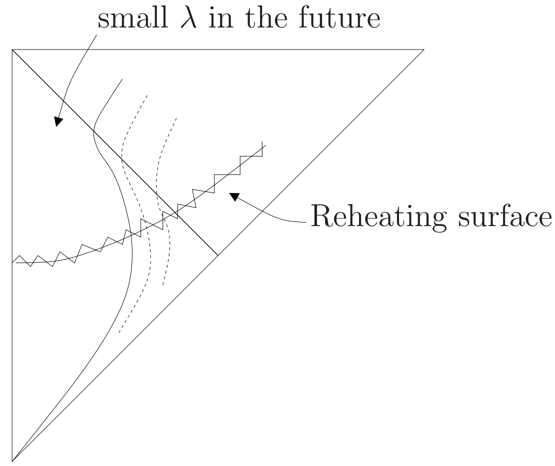
$$\phi \sim \frac{M_p}{a^{3/2}}\cos(mt + \delta) \tag{22.52}$$

This oscillating scalar field produces particles, eg. by couplings like

$$\mathcal{L} \ni g\phi\bar{\psi}\psi \tag{22.53}$$

to reheating! We can describes where reheating is very efficient , leading to the transfer of energy from ϕ to radiation in a few Hubble times, and reheating the universe to a temperature (see [12])

$$T_r \sim \sqrt{H_* M_p} \tag{22.54}$$

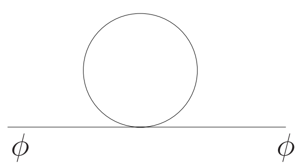


There could also be tensor fluctuations; recall that

$$\left(\frac{\delta\rho}{\rho}\right)_T \sim \sqrt{\epsilon}\left(\frac{\delta\rho}{\rho}\right)_S \sim 10^{-6} \quad \text{for } m^2\phi^2 \tag{22.55}$$

$\rightarrow$ Maybe visible to Planck!

A problem: radiative stability!



$\rightarrow \Lambda^2\phi^2$

Λ cutoff; so if $\Lambda \sim M_p$ we need $m_\phi \ll M_p$ – fine-tuning!

SUSY? But SUSY is broken in the early universe by the cosmological expansion itself! The natural cutoff is H, so the scalars like to aquire a mass

$$m \sim \mathcal{O}(H) \tag{22.56}$$

This would spoil slow-roll and ruin inflation. But, there may be symmetries we can use to protect scalar masses from radiative corrections.

axions, moduli, ...

Finally, the whole story works for an arbitrary potential

$$V = \frac{\lambda}{n}\phi^n \tag{22.57}$$

$$H = \sqrt{\frac{\lambda}{3n}}\frac{\phi^{n/2}}{M_p} \tag{22.58}$$

$$\dot{\phi} = -\sqrt{\frac{n\lambda}{3}} M_p \phi^{\frac{n}{2}-1} \tag{22.59}$$

slow roll

$$a = a_{\text{final}} e^{-\mathcal{N}} \tag{22.60}$$

$$\mathcal{N} = \frac{1}{2n}\left(\frac{\phi}{M_p}\right)^2 \tag{22.61}$$

Density contrast and tilt

$$\frac{\delta\rho}{\rho} = \frac{\sqrt{\lambda}}{2\pi\sqrt{3}n^{3/2}}\frac{\phi^{\frac{n}{2}+1}}{M_p^3} \tag{22.62}$$

$$n_s = 1 - \frac{n+2}{2\mathcal{N}} \tag{22.63}$$

Slow roll parameters:

$$\eta = -\frac{n(n-2)}{2}\frac{M_p^2}{\phi^2}, \qquad \epsilon \sim \frac{n^2}{2}\frac{M_p^2}{\phi^2} \tag{22.64}$$

22.2 Baysian versus frequentist (an aside)

When doing cosmology, often the experiments that we are doing cannot be repeated in any reasonable fashion. Given the most elementary notion of probability, which is we consider an infinite ensemble of systems and "probability of X" is interpreted as the fraction of the ensemble that has property X. If we only get to look at one member of the ensemble (say our universe), it is not clear that probabilistic statements mean anything at all!

While the above notion of probability is the most elementary (and the one usually employed in an undergraduate quantum mechanics class to assign some meaning to $|\Psi|^2$), it is not the only one. There are two main schools of thought on what probability means:

Bayesian The Bayesian approach treats probability as a degree of belief that something is true. If I am a weatherman, I feel 70% sure it is going to rain today. If I am the man on the street, I may only feel 40% sure it is going to rain.[1] The probability is subjective, so both the weatherman and the man on the street could be "right" – they have access to different amounts of information and have differing degrees of confidence that it will rain.

Frequentist This is a fancy name for the ensemble approach. Here, the probability really does exist – it is some number. In the rain case above, it is the probability would be very close to 1 or 0 (as the laws of atmospheric physics are basically deterministic on the time scales of a day). If we averaged over some initial positions and velocities, we could do this over entire ensemble and know the true probability that it rains on days "close" to today is 74%. Not suprisingly, the weatherman was closer! Note that the "true probability" does not even make sense in Bayesian statistics.

For one off "experiments", like the universe, the Bayesian approach may be the best that we can do.

Note some descriptions are better than others for different things. Think of the average age of people in Davis – there is a definite answer to the question. To *find* the answer, you may take a sample of people. After some analysis, the frequentist can give tell you "I am 95% sure that the true mean is contained in (30,44)" while the Bayesian approach would tell you "I believe that the mean is most likely between (30,44)".

Granted, these are issues of philosophy and mathematics. However, it is worth bearing them in mind as they may provide fundamental limitations to what can be done in cosmology (or at least indicate the reasons why there is a limited amount we can do).

[1] Degree of belief is hard to define, but usually it is done by betting odds. The probability is determined by if the person bet on it, at which odds would they have zero expectation value.

Chapter 23

Lecture 23

23.1 Some other variants of chaotic inflation

23.1.1 R+R^2 inflation

(Reference for this section is [22]).

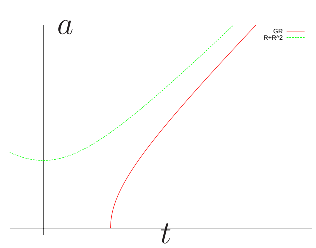
Consider Einstein's equations and include leading order quantum corrections

$$G_{\mu\nu} = \frac{1}{M_p^2}\left(T_{\mu\nu}^{(\mathrm{cl})} + \langle T_{\mu\nu}\rangle\right) \tag{23.1}$$

In general

$$\langle T_{\mu\nu}\rangle = a\left(2\nabla_\mu\nabla_\nu R - 2g_{\mu\nu}\nabla^2 R - 2RR_{\mu\nu} + \frac{1}{2}g_{\mu\nu}R^2\right) + \underbrace{b\mathcal{O}\left(R^2\right)}_{\text{non-local, etc.}}$$

Starobinsky's motivation was to argue that the initial singularity was avoided because quantum corrections forced the scale factor to "bounce" from zero!



But, for the bounce to occur $\ddot{a} > 0 \Rightarrow$ inflation in disguise.

Subsequently Starobinsky realised that his equations of motion can be derived from an action principle starting with

$$S = \int \mathrm{d}^4x\sqrt{g}\left(\frac{M_p^2}{2}R + \frac{\epsilon}{12}R^2 - \mathcal{L}_m\right) \tag{23.2}$$

However, the equations appear as a 4th order differential equations!

$$\nabla^2 R \propto \frac{\mathrm{d}^4}{\mathrm{d}t^4}g_{\mu\nu}, \qquad \text{etc.} \tag{23.3}$$

$\Rightarrow$ there are new degrees of freedom in the gravitational sector!

Recall: a degree of freedom is described by x, p or equivalently by 2nd order differential equations $\Rightarrow$ 2 integration constants x_0 and p_0 (per degree of freedom).

When the differential equations is 4th order $\Rightarrow$ 4 integration constants! x_0, p_0, y_0, $q_0 \rightarrow$ the old degree of freedom plus an extra degree of freedom.

This diagnostic procedure is quite general in fact.

There is quite a nice trick to diagonalise the metric in terms of all the degrees of freedom in the theory (see Brian Whitt [23]).

Define new metric $\tilde{g}_{\mu\nu}$ and a new scalar field $\tilde{\phi}$ such that

$$\tilde{g}_{\mu\nu} = \left(1 + \frac{\epsilon}{3}\frac{R}{M_p^2}\right)g_{\mu\nu} \tag{23.4}$$

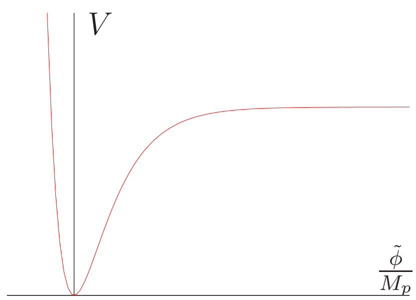
$$\tilde{\phi} = \sqrt{\frac{3}{2}}M_p \ln\left(1 + \frac{\epsilon}{3}\frac{R}{M_p^2}\right) \tag{23.5}$$

Then the action can be rewritten

$$S = \int d^4x \sqrt{\tilde{g}} \left(\frac{M_p^2}{2}\tilde{R} - \frac{1}{2}(\partial\tilde{\phi})^2 - V(\tilde{\phi}) \right) - \int d^4x \sqrt{\det \tilde{g}_{\mu\nu} \exp\left(-\sqrt{\frac{2}{3}} \frac{\tilde{\phi}}{M_p} \right)} \mathcal{L}\left(\psi, \tilde{g}_{\mu\nu} \exp\left(-\sqrt{\frac{2}{3}} \frac{\tilde{\phi}}{M_p} \right) \right) \tag{23.6}$$

⇒ Matter is *non-minimally* coupled directly to $\tilde{\phi}$.
⇒ Typical for supergravity theories with extra scalars.

$$V(\tilde{\phi}) = \frac{3}{4}\frac{M_p^4}{\epsilon}\left(1 - e^{-\sqrt{\frac{2}{3}}\frac{\tilde{\phi}}{M_p}}\right)^2 \tag{23.7}$$



⇒ a perfectly flat potential for $\phi \gg M_p$! → Sounds perfect for chaotic inflation.

The problem however is that the model needs $\epsilon \gg 1$ to have $V < M_p^4$! But this means that at large curvatures, $\epsilon R^2 > M_p^2 R$ – reliability?

Recently there were attempts to resurrect this idea in the brane world scenarios with AdS bulk.

$\epsilon \sim \#$ at hidden sector field theory degrees of freedom.
Large N CFT → $\epsilon \gg 1$. See Hawking, Hertog & Reall [15].

23.1.2 Power law inflation

(References for this section [24])

Suppose that the potential $V(\phi)$ is a little steeper; so $\dot{\phi}$ cannot be completely neglected during inflation. Now clearly, whether we have inflation or not depends on just how steep $V(\phi)$ is. In particular, note from

$$3H^2M_p^2 = \frac{\dot{\phi}^2}{2} + V(\phi) \tag{23.8}$$

That both $H = \frac{\dot{a}}{a}$ and $\dot{\phi}$ partake of $V(\phi)$. So steeper $V(\phi)$ means that $\dot{\phi}$ "steals" more $V(\phi)$ and so universe expands more slowly.

But from

$$w = \frac{p}{\rho} = \frac{\frac{\dot{\phi}^2}{2} - V}{\frac{\dot{\phi}^2}{2} + V} \tag{23.9}$$

we see that we can tolerate $\dot{\phi}$ as long as $w < -1/3$ and stays that way for a sufficiently long time.

Prototype example: exponential potentials. These are a nice benchmark for parameterising $n_s - 1$. So let

$$V = V_0 e^{c\frac{\phi}{M_p}}. \tag{23.10}$$

Then the field equations reduce to

$$3H^2 = \frac{1}{M_p^2}\left(\frac{\dot{\phi}^2}{2} + V_0 e^{c\frac{\phi}{M_p}}\right) \tag{23.11}$$

$$\ddot{\phi} + 3H\dot{\phi} + c\frac{V_0}{M_p}e^{c\frac{\phi}{M_p}} = 0 \tag{23.12}$$

Now, clearly, the qualitative properties of the solution CANNOT depend on V_0 and ϕ_0 separately. Reason: consider $\phi \to \phi + \phi_0$

$$V = V_0 \exp\left(c\frac{\phi}{M_p} + c\frac{\phi_0}{M_p}\right) \tag{23.13}$$

$$= \tilde{V}_0 \exp\left(c\frac{\phi}{M_p}\right), \qquad \text{where } \tilde{V}_0 = V_0 \exp\left(c\frac{\phi_0}{M_p}\right) \tag{23.14}$$

Thus, instead of parameterising the solutions by V_0 we can do it by ϕ_0. $\phi \to \phi + \phi_0$ a pseudosymmetry!

Now: one can show that this system has a late time attractor iff it is an accelerating cosmology with $\ddot{a} > 0$, or equivalently

$$a \sim t^p, p > 1$$

This works as long as

$$a = a_0 \left(\frac{t}{t_0}\right)^p \tag{23.15}$$

$$\phi = \phi_0 + \phi_1 M_p \ln\left(\frac{t}{t_0}\right) \tag{23.16}$$

Plug into the equations of motion

$$H = \frac{\dot{a}}{a} = \frac{p}{t} \tag{23.17}$$

$$\dot{\phi} = \frac{\phi_1}{t} M_p \tag{23.18}$$

$$3\frac{\phi^2}{t^2} = \frac{\phi_1^2}{2t^2} + \frac{V_0}{M_p^2} e^{\frac{c\phi_0}{M_p} + c\phi_1 \ln(t/t_0)} \tag{23.19}$$

$$\therefore 3\frac{p^2}{t^2} = \frac{\phi_1^2}{2t^2} + \nu\left(\frac{t}{t_0}\right)^{c\phi_1} \tag{23.20}$$

where $\nu = V_0 e^{c\phi_0/M_p}/M_p^2$. For the equation to hold at all times, we require $c\phi_1 = -2!$.

Now cancel t^{-2}:

$$3p^2 = \frac{\dot{\phi}^2}{2} + \vartheta t_0^2 \tag{23.21}$$

Next

$$\ddot{\phi} = -\frac{\phi_1 M_p}{t^2} \tag{23.22}$$

So

$$-\frac{\phi_1 M_p}{t^2} + \frac{3p\phi_1 M_p}{t^2} + c\frac{V_0}{M_p} e^{c\frac{\phi_0}{M_p}} \left(\frac{t_0}{t}\right)^2 = 0 \tag{23.23}$$

So, cancelling M_p/t^2:

$$\nu t_0^2 = \frac{\phi_1}{c} - \frac{3p\phi_1}{c} \tag{23.24}$$

Eliminate νt_0^2 from these two equations:

$$c\phi_1 = -2 \rightarrow \phi_1 = -\frac{2}{c} \tag{23.25}$$

$$3p^2 - \frac{\phi_1^2}{2} = \frac{\phi_1}{c} - \frac{3p\phi_1}{c} \tag{23.26}$$

Eliminate $\frac{1}{c} = -\frac{\phi_1}{2}$ so then

$$3p^2 - \frac{\phi_1^2}{2} = -\frac{\phi_1^2}{2} + \frac{3p}{2}\phi_1^2 \tag{23.27}$$

$$p = \frac{\phi_1^2}{2} = \frac{2}{c^2} \tag{23.28}$$

Thus

$$a = a_0 \left(\frac{t}{t_0}\right)^{\frac{2}{c^2}} \tag{23.29}$$

$$\phi = \phi_0 - \frac{2}{c} M_p \ln\left(\frac{t}{t_0}\right) \tag{23.30}$$

and $\ddot{a} > 0$ (i.e. p¿1) when

$$c^2 < 2 \tag{23.31}$$

Note: this is why it is difficult to get inflation from compactifiation of supergravities; usually one gets $c^2 > 6$ instead of $c^2 < 2$!

What is the spectral index?

$$\frac{\delta\rho}{\rho} = \frac{1}{2\pi}\frac{H^2}{\dot{\phi}} = \frac{p^2}{|\phi_1|}\frac{1}{M_p t} \tag{23.32}$$

where $p = 2/c^2$, $|\phi_1| = 2/c$. So

$$\frac{\delta\rho}{\rho} = \frac{2}{c^3}\frac{1}{M_p t} \tag{23.33}$$

Horizon crossing condition $\lambda = \frac{a}{k} = \frac{1}{H}$. This implies (as always)

$$k = aH = k_0 t^{p-1} \tag{23.34}$$

so

$$\frac{\delta\rho}{\rho} = \left(\frac{\delta\rho}{\rho}\right)_0 k^{\frac{1}{1-p}} \tag{23.35}$$

Hence:

$$\ln\frac{\delta\rho}{\rho} = \frac{1}{1-p}\ln k + A \tag{23.36}$$

and

$$n_s = 1 + 2\frac{\mathrm{d}\ln\frac{\delta\rho}{\rho}}{\mathrm{d}\ln k} = 1 + \frac{2}{1-p} \tag{23.37}$$

Thus with $p = 2/c^2$,

$$n_s = 1 - \frac{2}{p-1}. \tag{23.38}$$

or

$$n_s = 1 - \frac{2c^2}{2-c^2} \tag{23.39}$$

From $n_s = 1 \pm 0.05$ we find that p must be pretty large, $p > \frac{2}{0.05} + 1 = 41$.

So the slow roll condition $\epsilon = \frac{3\dot{\phi}^2}{2V} \ll 1$ must be well satisfied!

Chapter 24

Lecture 24

24.1 Modular inflation

(A list of references have been collected in the bibliography under [21])

Moduli: flat directions in SUSY models → weakly coupled particles whose masses all vanish in the SUSY limit.

Good candidates for the inflation ⇒ we need to have light scalars whose masses are small, and remain small even after the quantum corrections are accounted for. Moduli fields can achieve this because they are protected by symmetries.

They are generally arising in super-gravity compactifications as fields that parameterise size and shape of the extra dimensions.

General idea: SUSY breaking (or compactification and stabilisation of the extra dimension(s)) gives rise to a potential for the modulus ϕ, where there was no potential at higher scales. This is going to be of the form

$$V = M^4 \mathcal{O}\left(\frac{\phi}{\Lambda}\right) \tag{24.1}$$

where

- M: symmetry breaking scale. Say $M \sim M_{\mathrm{GUT}} \sim 10^{16}\,\mathrm{GeV}$

- Λ: cutoff. Say $\Lambda \sim M_p \sim 10^{19}\,\mathrm{GeV}$
- $\mathcal{O}\left(x\right)$ is some (analytical) dimensionless fuction

$\Rightarrow$ all the scales during inflation are determined by the parameters M and Λ.

Note: from the Friedmann equation, if V admits the slow roll regime

$$H^2 \sim \frac{V}{M_p^2} \sim \frac{M^4}{M_p^2} \tag{24.2}$$

so $H \sim M^2/M_p$.

But: recall that the existence of the slow roll regime requires that

$$m \ll H \tag{24.3}$$

where $m^2 = \partial^2 V/\partial\phi^2$. Taking $\Lambda \sim M_p$, $V = M^4\mathcal{O}\left(\phi/M_p\right)$ and expanding $\mathcal{O}$ we set (ignoring the linear term)

$$V = M^4 + c\frac{M^4}{M_p^2}\phi^2 \tag{24.4}$$

so

$$m \sim \sqrt{C}\frac{M^2}{M_p} \sim \sqrt{C}H \tag{24.5}$$

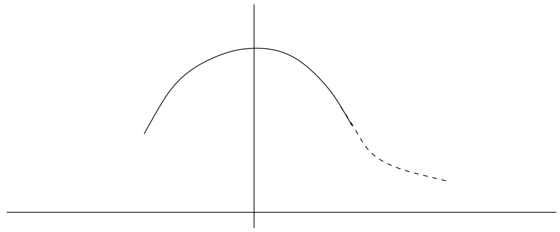
Thus, we must have $\sqrt{C} \ll 1 \Rightarrow C \ll 1$.

This is the price we pay: we need a small parameter built into the potential. Essentially this is always the case for inflation:

$$N \sim \frac{1}{\epsilon} \rightarrow a_{\text{final}} \sim a_0 e^{1/\epsilon} \tag{24.6}$$

Onto the specifics (see [13]):
Consider the potential to be

$$\mathcal{O} = 1 - \left(\frac{\phi}{\mu}\right)^2 \tag{24.7}$$



where inflation takes place. In the slow roll regime, we have

$$3H^2 = \frac{M^4}{M_p^2}\left(1-\left(\frac{\phi}{\mu}\right)^2\right) \tag{24.8}$$

$$3H\dot{\phi} = 2M^4\frac{\phi}{\mu^2} \tag{24.9}$$

$$\Rightarrow H = \frac{M^2}{\sqrt{3}M_p}\sqrt{1-\left(\frac{\phi}{\mu}\right)^2} \tag{24.10}$$

$$\dot{\phi} = \frac{2M^2M_p^2}{\sqrt{3}\mu^2}\frac{\phi}{\sqrt{1-(\phi/\mu)^2}} \tag{24.11}$$

Recall that

$$\epsilon = M_p^2\frac{(\partial_\phi V)^2}{2V^2} \tag{24.12}$$

$$\eta = -\epsilon + M_p^2\frac{\partial_\phi^2 V}{V} \tag{24.13}$$

$\rightarrow M^4$ cancels out! So

$$\epsilon = M_p^2\frac{(\partial_\phi \mathcal{O})^2}{2\mathcal{O}^2} = \frac{2M_p^2\phi^2}{\mu^4} \tag{24.14}$$

$$\eta = -\epsilon + M_p^2\frac{\partial_\phi^2 \mathcal{O}}{\mathcal{O}} = \frac{2M_p^2}{\mu^2} \tag{24.15}$$

The condition $\eta \ll 1 \Rightarrow \mu^2 \gg 2M_p^2$.

That is the fine-tuning! note also that for small $\phi < \mu$, $\epsilon < \eta$. So η is what controls the validity of the slow roll regime!

Now, the number of e-folds is

$$N = \ln\left(\frac{a}{a_0}\right) = \int_{t_0}^{t} \mathrm{d}t H = \int_{\phi_0}^{\phi} \mathrm{d}\phi \frac{H}{\dot{\phi}} \tag{24.16}$$

Plugging in the slow roll equations

$$N \approx \frac{1}{\eta}\left(\ln\frac{\phi}{\phi_0} + \frac{\phi_0^2 - \phi^2}{2\mu^2}\right) \tag{24.17}$$

During inflation, ϕ does not change that much, and so the total number of e-folds is basically controlled by η. To have $N \gtrsim 70$ we need $\eta \lesssim 1/70$.

Inflation ends when ϕ reaches $\phi \sim \mu$ (Note $\mathcal{O} = 1 - (\phi/\mu)^2 \sim 0$ when $\phi \sim \mu$). We define again

$$\mathcal{N} = \bar{N} - N \tag{24.18}$$

as the counter of e-folds before the end of inflation and find

$$\mathcal{N} = \frac{1}{\eta}\left[\ln\frac{\mu}{\phi} + \frac{\phi^2 - \mu^2}{2\mu^2}\right] \tag{24.19}$$

$$H = \frac{M^2}{\sqrt{3}M_p}\sqrt{1 - \left(\frac{\phi}{\mu}\right)^2} \tag{24.20}$$

$$a = a_{\text{final}} e^{-\mathcal{N}} \tag{24.21}$$

Then:

$$\frac{\delta\rho}{\rho} = \frac{H^2}{2\pi\dot{\phi}} = \frac{1}{2\sqrt{3}\pi}\frac{M^2}{M_p^3}\frac{\mathcal{O}^{3/2}}{\partial_\phi \mathcal{O}} \tag{24.22}$$

RHS is a function of ϕ, eliminate it in favour of k using

$$\lambda_{\text{wavelength}} = \frac{a}{k} = \frac{1}{H} \to k = aH \tag{24.23}$$

This yields $k \approx k_0 e^{-\mathcal{N}}$.

So

$$\frac{\delta\rho}{\rho} \approx \frac{2\mu M^2}{8\pi\sqrt{3}M_p^3}\left(\frac{k_0}{k}\right)^{\eta}\left[1-\left(\frac{k}{k_0}\right)^{2\eta}\right]^{3/2} \tag{24.24}$$

To set

$$\left.\frac{\delta\rho}{\rho}\right|_{50} \sim 5\times 10^{-5}$$

we need $M \approx 8\times 10^{-3}M_p$ – GUT scale!

$$n_s = 1 + 2\frac{\mathrm{d}\ln\frac{\delta\rho}{\rho}}{\mathrm{d}\ln k} = 1 - 2\eta - \frac{6\eta}{\left(\frac{k_0}{k}\right)^{2\eta}-1} \tag{24.25}$$

50 e-folds before the end, $n_s \approx 0.95$.

24.1.1 More general cases

Consider the generalisation of the above:

$$\mathcal{O} = 1 - \left(\frac{\phi}{\mu}\right)^n, \qquad n > 2 \tag{24.26}$$

Then:

$$H = \frac{M^2}{\sqrt{3}M_p}\sqrt{1-\left(\frac{\phi}{\mu}\right)^n} \tag{24.27}$$

$$\dot{\phi} = \frac{nM^2M_p}{\sqrt{3}\mu^n}\frac{\phi^{n-1}}{\sqrt{1-\left(\frac{\phi}{\mu}\right)^n}} \tag{24.28}$$

$$a = a_{\mathrm{final}}e^{-\mathcal{N}} \tag{24.29}$$

$$\mathcal{N} = \frac{\mu^n}{n(n-2)M_p^2\phi^{n-2}} + \frac{(n-4)\mu^2}{2n(n-2)M_p^2} \tag{24.30}$$

and finally

$$\frac{\delta\rho}{\rho} = \frac{1}{2\pi n\sqrt{3}}\left(\frac{M}{M_p}\right)^2\frac{\phi}{M_p}\left(\frac{\mu}{\phi}\right)^2\left[1-\left(\frac{M}{M_p}\right)^n\right]^{3/2} \tag{24.31}$$

Numerically:

$$\left.\frac{\delta\rho}{\rho}\right|_{50} \sim 5\times 10^{-5} \tag{24.32}$$

$$\Rightarrow M \sim 4\times 10^{-2}\sqrt{n}(n-2)^{1/4}M_p \tag{24.33}$$

or,

$$H \sim \text{few}\times 10^{14}\sqrt{n}\sqrt{n-2}\,\text{GeV} \tag{24.34}$$

But

$$\left(\frac{\delta\rho}{\rho}\right)_T \sim \frac{H}{M_p} < 10^{-6} \tag{24.35}$$

→ so tensor fluctuations have not been directly observed yet!

Thus, as an experimental constraint

$$H < 10^{-6}M_p \tag{24.36}$$

$$\text{i.e.}\quad H < 7\times 10^{13}\,\text{GeV} \tag{24.37}$$

Therefore, $n > 2$ has already been ruled out!

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