Lecture 4a

Wednesday, January 15, 2020 4:17 PM

The "tensors"

$\delta^\mu{}_\nu$, $\eta_{\mu\nu}$, $\eta^{\mu\nu}$, $\tilde{\varepsilon}_{\mu\nu\alpha\beta}$

$\underbrace{\delta^\mu{}_\nu,\ \eta_{\mu\nu},\ \eta^{\mu\nu}}_{\text{Tensors}}$ $\underbrace{\tilde{\varepsilon}_{\mu\nu\alpha\beta}}_{\text{Tensor density}}$

Tensors Tensor density

$\boxed{\text{For each of these tensors, their components are the same in an inertial frame in flat spacetime.}}$ *

$$\delta^\mu{}_\nu = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$$

Identity

used to define inverse

$$A^{\mu\nu} B_{\nu\rho} = \delta^\mu{}_\rho$$

$$\Rightarrow B_{\nu\rho} = \left(A^{\mu\nu}\right)^{-1}$$

$\eta_{\mu\nu}$ defines invariant

(0,2)

$$(\Delta S)^2 = \eta_{\mu\nu} \Delta X^\mu \Delta X^\nu$$

(0,0)

$$= \eta_{\mu'\nu'} \Delta X^{\mu'} \Delta X^{\nu'}$$

same η

Note: Any (0,2) tensor $A_{\mu\nu}$ will give a scalar $A_{\mu\nu} \Delta X^\mu \Delta X^\nu$

will give a scalar $A_{\mu\nu}\Delta x^\mu \Delta x^\nu$

but in general the components of A will be different in different frames.

<u>Also</u> by convention, $\eta_{\mu\nu}$ is used to "lower an index"

Given $A^{\mu\nu}$, $\quad A^{\mu}{}_{\nu} \equiv \eta_{\nu\rho} A^{\mu\rho}$

$\quad\quad (2,0) \quad\quad\quad (1,1)$

Again, <u>any</u> (0,2) tensor can be used to convert a (0,2) to a (1,1), but $\eta_{\mu\nu}$ is special because it looks the same in all frames.

(Note: $\delta^{\mu}{}_{\nu}$ is a (1,1) rank tensor, so it is not a contender)

$\eta_{\mu\nu}$ also defines the "dot" or "scalar" product:

$$V \cdot W \equiv \eta_{\mu\nu} V^\mu W^\nu$$

and thus the <u>norm</u>

<u>Norm of V:</u>

$$\eta_{\mu\nu} V^\mu V^\nu \begin{cases} <0 & V^\mu \text{ is timelike} \\ =0 & \text{" " lightlike (null)} \end{cases}$$

$$\eta_{\mu\nu} V^\mu V^\nu \begin{cases} <0 & V \text{ is timelike} \\ =0 & \text{" " lightlike (null)} \\ >0 & V^\mu \text{ is spacelike} \end{cases}$$

Scalar ⇒ invariant under Lorentz tfm's.

Note: Norm of $X = (1,1,0,0) = 0$!

$\eta^{\mu\nu}$ The inverse of $\eta_{\mu\nu}$:

(2,0) $\eta_{\mu\nu}\eta^{\nu\rho} = \delta_\mu^\rho$

Used for raising an index

(only (2,0) the same in all frames.)

$\tilde{\varepsilon}_{\mu\nu\rho\sigma}$ "Levi-Civita Symbol"

The "completely antisymmetric with $\tilde{\varepsilon}_{12\ldots n} = 1$

$\tilde{\varepsilon}_{12}$

In 2d: $\tilde{\varepsilon}_{ij} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

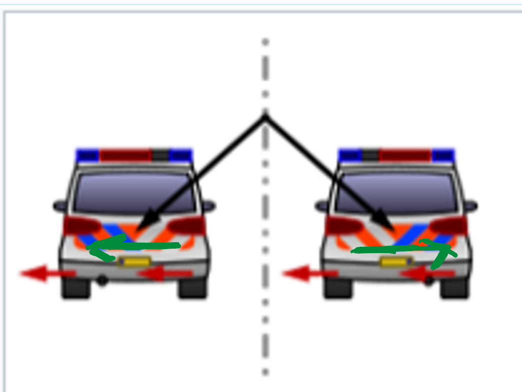
In 3d harder to write out,

In 3d harder to write out, but we know it from the cross product:

$$\begin{aligned}\mathbf{a}\times\mathbf{b} = \quad & a_1b_1\mathbf{0} + a_1b_2\mathbf{k} - a_1b_3\mathbf{j} \\ & - a_2b_1\mathbf{k} + a_2b_2\mathbf{0} + a_2b_3\mathbf{i} \\ & + a_3b_1\mathbf{j} - a_3b_2\mathbf{i} + a_3b_3\mathbf{0} \\ = & (a_2b_3 - a_3b_2)\underset{\vec{e}_1}{\mathbf{i}} + (a_3b_1 - a_1b_3)\underset{\vec{e}_2}{\mathbf{j}} + (a_1b_2 - a_2b_1)\underset{\vec{e}_3}{\mathbf{k}}\end{aligned}$$

$$= \vec{e}_i \, a_j \, b_k \, \hat{\varepsilon}_{ijk}$$

a×b is a <u>pseudovector</u>



Each wheel of the car on the left driving away from an observer has an angular momentum pseudovector pointing left. The same is true for the mirror image of the car. The fact that the arrows point in the same direction, rather than being mirror images of each other indicates that they are pseudovectors.

$\Rightarrow$ $\tilde{\varepsilon}_{ijk}$ is <u>pseudotensor</u>

(if it were a real (.) tensor,

$\varepsilon_{ijk} A^j B^k$ would be a

real vector

<u>Note</u> $\tilde{\varepsilon}$ has the same number of indices as the dimension of space.

$\Rightarrow$ 4 for spacetime

$\Rightarrow$ "No $\times$ product in 4d"

(but can generalize by contracting $\tilde{\varepsilon}$ with three vectors A^μ, B^ν, C^ρ