The "tensors"

For each of these tensors, their components are the same in an inertial frame in flat spacetime.

\[
\delta^\mu_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]

Identity used to define inverse

\[
A^{\mu\nu} B_{\nu\rho} = \delta^\mu_\rho
\]

\[
\Rightarrow B_{\nu\rho} = \left( A^{\mu\nu} \right)^{-1}
\]

\[
\mathcal{N}_{\mu\nu}
\]

defines invariant

\[
(\Delta S)^2 = \mathcal{N}_{\mu\nu} \Delta x^\mu \Delta x^\nu
\]

\[
(0,0)
\]

\[
\overset{\text{same } \mathcal{N}}{=} \mathcal{N}_{\mu\nu} \Delta x^\mu \Delta x^\nu.
\]

Note: Any \((0,2)\) tensor \(A^{\mu\nu}\) will give a scalar \(A^{\mu\nu} \Delta x^\mu \Delta x^\nu\)
will give a scalar $A_{\mu\nu} \cdot dx \cdot dx$
but in general the components of $A$ will be different in different frames.

Also by convention, $\nabla_\mu$ is used to "lower an index"

Given $A^{\mu\nu}$, $A_\mu \equiv \eta_{\mu\nu} A^{\nu\rho}$

$(2,0)$ $(1,1)$

Again, any $(0,2)$ tensor can be used to convert a $(0,2)$ to a $(1,1)$, but $\nabla_\mu$ is special because it looks the same in all frames.

(Note: $\nabla_\mu$ is a $(1,1)$ rank tensor, so it is not a contender)

$\nabla_\mu$ also defines the "dot" or "scalar" product:

$V \cdot W = \eta_{\mu\nu} V^\mu W^\nu$

and thus the norm

Norm of $V$:

$\eta_{\mu\nu} V^\mu V^\nu$ $\neq 0$ $V^\mu$ is timelike

$\eta_{\mu\nu} V^\mu V^\nu = 0$ " lightlike" (null)
\[ \nu^\mu \nu_\mu \begin{cases} < 0 & \text{tense} \\ = 0 & \text{lightlike (null)} \\ > 0 & \text{space-like} \end{cases} \]

Scalar: invariant under Lorentz transform.

Note: Norm of \( X = (1,1,0,0) = 0 \).

\( \nabla^\mu \) The inverse of \( \nu^\mu \):

\( (2,0) \quad \nabla^\mu \nabla^\nu \equiv \delta^\mu_\nu \)

Used for raising an index
(only \( (2,0) \) the same in all frames)

\( \tilde{\epsilon}_{\alpha \nu \rho \sigma} \) "Levi-Civita Symbol"

The "completely" antisymmetric with \( \tilde{\epsilon}_{12 \cdots n} = 1 \)

In 2d: \( \tilde{\epsilon}^i_{\ j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \)

In 3d, harder to write out.
In 3d harder to write out, but we know it from the cross product:

\[
\mathbf{a} \times \mathbf{b} = a_1 b_1 \mathbf{i} + a_1 b_2 \mathbf{j} - a_1 b_3 \mathbf{k} - a_2 b_1 \mathbf{i} - a_2 b_2 \mathbf{j} + a_2 b_3 \mathbf{k} + a_3 b_1 \mathbf{i} + a_3 b_2 \mathbf{j} - a_3 b_3 \mathbf{k} = (a_2 b_3 - a_3 b_2) \mathbf{i} + (a_3 b_1 - a_1 b_3) \mathbf{j} + (a_1 b_2 - a_2 b_1) \mathbf{k}
\]

\[
\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3
\]

\[
\mathbf{a} \times \mathbf{b} = \hat{\mathbf{e}}_i a_j b_k \epsilon_{ijk}
\]

\(\mathbf{a} \times \mathbf{b}\) is a pseudovector
\[ E_{ijk} \text{ is pseudotensor} \]

(If it were a real tensor, \[ E_{ijk} A^{i} B^{j} \] would be a real vector.)

Note: \( E \) has the same number of indices as the dimension of space.

\[ \Rightarrow 4 \text{ for spacetime} \]

\[ \Rightarrow \text{No } \times \text{ product in 4d} \]

(But can generalize by contracting \( E \) with three vectors \( A^i, B^j, C^k \)).