Notes on Dimensional Analysis
Physics 262
Early Universe Cosmology
A. Albrecht
Updated January 8, 2007

Contents

1 Notes on Dimensional Analysis and the Fundamental Constants 1

1.1 Powers of energy ("energy units") 2

1.2 Powers of Length ("Geometrized" units) 4

2 Further comments (optional) 4

1 Notes on Dimensional Analysis and the Fundamental Constants

There are a number of ways of using the fundamental constants to change the dimensions of a quantity.

1.1 Powers of energy ("energy units")

For example. Any physical quantity (except dimensionless numbers) can be multiplied by suitable powers of $\hbar$, c and k_B to produce a quantity with dimensions of energy to some power.

This table gives some key examples:

unit:	multiplied by:	gives:
length l	$(\hbar c)^{-1}$	$(\text{energy})^{-1}$
time t	$(\hbar)^{-1}$	$(\text{energy})^{-1}$
mass m	c^2	$(\text{energy})^{+1}$
temperature T	k_B	$(\text{energy})^{+1}$

One can regard this transformation as a conversion to "energy units". When you convert the fundamental constants themselves into energy units one finds that "$\hbar = c = k_B = 1$". Sometimes energy units are simply called "the units where $\hbar = c = k_B = 1$".

Energy units may not always be of practical use, but it does turn out to be useful when doing particle physics, and when studying the very early universe. The practical reason is that many of the equations become simpler. I try to describe the physical reason for this below:

The matter in the very early universe consisted of a smooth distribution of fundamental particles moving about thermally at relativistic speeds. Under these circumstances, essentially all dimensional quantities can be characterised by the thermal energy $k_B T$.

For example, the only way the concept of *length* is relevant is when considering the wavelength of a typical particle. There is essentially no other length scale in the problem. A typical particle has energy $\approx k_B T$, and relativistic particles (ie photons) with energy k_BT have wavelengths given by $(2\pi)(k_BT)^{-1}(\hbar c)$. (NB: What follows are rough estimates, in which we do not expect to get factors of 2π right!)

Likewise, one can start with the energy k_BT, and work from right to left in the table to get characteristic values of time and mass. (Starting with $(k_BT)^{-1}$ and *dividing* by $\hbar^{-1}$ gives, $\hbar/(k_BT)$, which is the amount of *time* required for a particle with energy k_BT to move one (wavelength)/(2π).)

From these characteristic quantities one can build others: Momentum has units of (mass)×(length)×(time)$^{-1}$. Plugging in the characteristic values gives:

$$\left(\frac{k_BT}{c^2}\right) \times \left(\frac{\hbar c}{k_BT}\right) \times \left(\frac{\hbar}{k_BT}\right)^{-1} = k_BT/c, \tag{1}$$

the momentum of a relativistic particle with energy k_BT.

As another example, let us calculate the characteristic quantity of energy density: Energy density has units of (energy)×(length)$^{-3}$. Using the characteristic quantities for each of these gives

$$\frac{k_BT}{[(\hbar c)/(k_BT)]^3} = \left(\frac{k_B^4}{\hbar^3 c^3}\right) T^4. \tag{2}$$

Note how close this is to the energy density u_γ of photons in thermal equilibrium:

$$u_\gamma = \frac{\pi^2}{15}\left(\frac{k_B^4}{\hbar^3 c^3}\right) T^4 \tag{3}$$

The fact the energy units are useful for estimating quantities in the early universe is a reflection of the particular state of matter at that time. (See more about this is Section 2)

1.2 Powers of Length ("Geometrized" units)

One can also leave out $\hbar$ and instead use G (with c and k_B) to convert all units to powers of length.

unit:	multiplied by:	gives:
energy	G/c^4	length
time t	c	length
mass m	G/c^2	length
temperature T	$G/(c^4 k_B)$	length

These conversions are useful when gravity is producing the dominant effects (eg for studying the growth of inhomogeneities due to gravitational collapse). Again, certain equation are made simpler in these units, and this is a reflection of the fact that the universe is in a particular state, with certain effects (eg gravitational collapse) governing its evolution.

2 Further comments (optional)

You may find the following remarks of interest, but they are not part of the required material for this course.

The "fundamental constants" can be thought of as a conversion constants which convert one type of units to another.

For example the Boltzmann constant, k_B, converts Kelvins to Joules. The Boltzmann constant is considered more "fundamental" than the conversion factor between inches and meters because it was a fundamental discovery that Kelvins and Joules actually measured the same quantity. People measured and discussed temperatures long before they realized that temperature is just a measure of energy. (It measures the energy in a typical degree of freedom of a large thermalized system.)

Likewise, $\hbar$ converts $(meters)^{-1}$ to units of momentum. People measured and discussed momentum long before quantum mechanics taught us to think of momentum in terms of the d/dx operator, which has units of inverse length.

Also, special relativity has taught us how space and time "rotate" into one another when one changes reference frames. The speed of light, c, can be thought of as conversion constant which converts units of time into units of space.

One's choice of units is always a matter of convenience. An astronomer measuring clusters of galaxies would much rather use Megaparsecs than inches. One's choice of units usually says a lot about the system one is looking at. Each of the fundamental constants mentioned above is related to an important property of our physical world.

The existence of k_B reflects the fact that it is usually convenient for us to measure thermal and macroscopic energies in different units. It just so happens that we live in a world where these energies typically have very different values.

Similarly, c reflects the fact that most matter around us is non-relativistic, and $\hbar$ exists because the inverse length scales corresponding to macroscopic momenta are very much smaller than the macroscopic length scales we are used to dealing with. (It is no accident that the fundamental constants are all very different from unity. If typical thermal energies were the same as macroscopic energies, "temperature" and "energy" would never have developed as separate notions. Similarly for $\hbar$ and c.)

The matter in the very early universe consisted of a smooth distribution of fundamental particles moving about at relativistic speeds. Compared with our familiar physical world, this is an extremely unusual state of matter. Except when discussing the universe as a whole, there are typically no separate "macroscopic" and "microscopic" worlds. Thus it is convenient to measure temperature in units of energy, and momentum in units of inverse length. In addition, the relativistic particle speeds make it convenient to measure space in units of time. One can thus say one is using units where "$\hbar = c = k_B = 1$". With this choice of units, the conversions which these constants perform are trivial! With these constants set to unity, one can freely multiply and divide by them and one can measure all quantities in units of energy to some power (as discussed above).